

Sharp Positivity Tests for Polynomial Hermitian Kernels

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Abstract

Let V be an r -dimensional space of complex polynomials and let $\mathcal{P}_k$ denote positivity on every k -point evaluation matrix. We prove that a Hermitian kernel supported on V is a holomorphic squared norm whenever it belongs to $\mathcal{P}_k$ and $r \leq 2k$. For every one-variable polynomial space, the least sufficient evaluation size is exactly $\lceil r/2 \rceil$, independently of its degrees. An indefinite polynomial kernel in $\mathcal{P}_k$ must consequently have at least $2k$ positive eigenvalues. A Hankel moment construction gives explicit sharp examples of signature $(2k, 1)$; for $k \geq 2$ their squares are squared norms. Combining the rank bound with the established Cauchy–Schwarz necessity for squared-norm roots gives a complete criterion at coefficient rank at most four: a nonzero real-valued polynomial has a squared-norm power precisely when its coefficient matrix is positive or negative semidefinite. This includes all one-variable polynomials of bidegree at most $(3, 3)$. The proof uses classical polynomial-secant geometry and constructibility. It does not classify squared-norm powers at arbitrary rank.

1 Positivity classes and the results

Let $V \subset \mathbb{C}[z_1, \dots, z_n]$ have dimension $r \geq 1$, and choose a basis $f = (f_1, \dots, f_r)^T$. For a Hermitian matrix H , set

$$K_H(z, \bar{w}) = f(w)^* H f(z).$$

The function is holomorphic in z and antiholomorphic in w . We say $K_H \in \mathcal{P}_k$ if every matrix $(K_H(z_i, \bar{z}_j))_{i,j=1}^k$ is positive semidefinite. Equivalently,

$$\left(\sum_{i=1}^k a_i f(z_i) \right)^* H \left(\sum_{i=1}^k a_i f(z_i) \right) \geq 0 \quad (a_i \in \mathbb{C}, z_i \in \mathbb{C}^n). \quad (1)$$

The harmless transpose between the two matrix conventions preserves positive semidefiniteness. Zero weights and repeated points are allowed.

The evaluation vectors span $\mathbb{C}^r$, since a linear functional vanishing on all of them would be a zero polynomial in V . Thus $K_H \in \mathcal{P}_\infty := \bigcap_k \mathcal{P}_k$ if and only if $H \succeq 0$. This is also equivalent to the diagonal being a finite sum of squared moduli of holomorphic polynomials. Independence of the products $f_i(z)f_j(w)$ makes the coefficient form unique.

Following D’Angelo–Varolin [2], the *stability index* $I(V)$ is the least k such that every Hermitian kernel supported on V belonging to $\mathcal{P}_k$ belongs to $\mathcal{P}_\infty$. The index is independent of the choice of basis.

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Theorem 1. *For every polynomial space V of dimension r ,*

$$I(V) \leq \left\lceil \frac{r}{2} \right\rceil.$$

If V consists of polynomials in one complex variable, equality holds. In particular, a polynomial Hermitian kernel of coefficient rank at most $2k$ belonging to $\mathcal{P}_k$ is a squared norm.

Here and below, coefficient rank is the rank of the unique polarized coefficient matrix, not the number of monomials displayed in an expression.

Theorem 2. *If a polynomial Hermitian kernel belongs to $\mathcal{P}_k$, then either it is a squared norm or its coefficient matrix has at least $2k$ positive eigenvalues. The bound is sharp for each $k \geq 1$.*

The positivity classes and their connection with Hermitian metrics are established in [2]. Its Theorem 1 gives the bound $k+1$ for an indefinite entire Hermitian function in $\mathcal{P}_k$, without assuming a polynomial kernel; its Theorem 3 computes stability for a particular binary family. Our statements concern polynomial kernels, and the algebraic image argument is essential to the improvement. We do not assert the $2k$ bound for arbitrary entire functions.

The geometric ingredient is the classical expected dimension of secants of a rational curve, proved here by a Wronskian calculation. Constructibility is supplied by Chevalley's theorem [5]. Thus the argument does not introduce a new positivity class or a new secant-dimension theorem. We also give explicit sharpness certificates and an application to the self-power question posed by D'Angelo in the AIM workshop [1, Squared norms, Problem 2].

2 Weighted polynomial evaluations

Write $C_k(f)$ for the complex Zariski closure of the image of

$$F_k : \mathbb{C}^k \times (\mathbb{C}^n)^k \longrightarrow \mathbb{C}^r, \quad F_k(a, z) = \sum_{i=1}^k a_i f(z_i).$$

It is an irreducible closed complex cone. The polynomial image is constructible by Chevalley's theorem and contains a nonempty Zariski-open subset of its closure. Such a subset is Euclidean dense in this complex irreducible variety. Consequently $C_k(f)$ is also the Euclidean closure of the image, and

$$K_H \in \mathcal{P}_k \iff x^* H x \geq 0 \text{ for every } x \in C_k(f). \quad (2)$$

An arbitrary Zariski-dense set need not be Euclidean dense; constructibility is what justifies this step.

Lemma 3. *For independent one-variable polynomials $f_1, \dots, f_r$,*

$$\dim_{\mathbb{C}} C_k(f) = \min(2k, r).$$

Proof. First suppose $r = 2k$. Independent polynomials have a nonzero Wronskian. For completeness, change basis to polynomials with distinct leading degrees $d_1 < \dots < d_{2k}$. In the leading term of their Wronskian the determinant of the falling factorials in the d_i is $\prod_{i < j} (d_j - d_i)$, up to nonzero leading coefficients. Choose $a \in \mathbb{C}$ where the Wronskian does not vanish.

At weights all equal to one, the Jacobian of F_k has columns

$$f(t_1), f'(t_1), \dots, f(t_k), f'(t_k).$$

For distinct fixed numbers b_i , put $t_i = a + \epsilon b_i$. Taylor expansion and the confluent Vandermonde determinant give

$$\begin{aligned} & \det(f(t_1), f'(t_1), \dots, f(t_k), f'(t_k)) \\ &= \epsilon^{2k(k-1)} \frac{\det(f(a), f'(a), \dots, f^{(2k-1)}(a))}{\prod_{j=0}^{2k-1} j!} \prod_{i < j} (b_j - b_i)^4 + O(\epsilon^{2k(k-1)+1}). \end{aligned} \quad (3)$$

The exponent comes from $\sum_{j=0}^{2k-1} j$ and the subtraction of one for each of the k derivative columns. The leading coefficient is nonzero. The map F_k therefore has a full-rank Jacobian and is dominant onto $\mathbb{C}^{2k}$.

If $r < 2k$, extend the independent polynomials to $2k$ independent polynomials and project the resulting dominant map onto the original coordinates. If $r > 2k$, project onto any $2k$ independent coordinate polynomials; the projected image has dimension $2k$. The original parameter space has dimension $2k$, giving the opposite bound. $\square$

Lemma 4. *If $r \leq 2k$ and $f_1, \dots, f_r$ are independent polynomials on $\mathbb{C}^n$, then $C_k(f) = \mathbb{C}^r$.*

Proof. Choose r points $z_1, \dots, z_r$ with independent evaluation vectors. Coordinatewise Lagrange interpolation gives a polynomial curve $\gamma : \mathbb{C} \rightarrow \mathbb{C}^n$ passing through those points at distinct parameters. The polynomials $f_j \circ \gamma$ remain independent, since their evaluation matrix at those parameters is invertible. Lemma 3 says that the closure of their weighted k -point sums is $\mathbb{C}^r$. Those sums are among the sums defining $C_k(f)$. $\square$

3 The stability and signature bounds

Proof of Theorem 1. For $r \leq 2k$, Lemma 4 and (2) show that $x^* H x \geq 0$ on all of $\mathbb{C}^r$, so $H \succeq 0$.

For sharpness in one variable, suppose $2k < r$. Lemma 3 makes $C_k(f)$ a proper closed cone. Choose a unit vector $v \notin C_k(f)$. Its compact unit slice satisfies

$$0 < \rho := \max_{\substack{x \in C_k(f) \\ \|x\|=1}} |v^* x|^2 < 1.$$

The upper inequality follows from equality in Cauchy–Schwarz: a maximizing vector with value one would be proportional to v . The lower inequality follows because the evaluation vectors span $\mathbb{C}^r$. Choose $1 < \alpha \leq 1/\rho$. Then $H = I - \alpha v v^*$ is indefinite but nonnegative on $C_k(f)$, hence $K_H \in \mathcal{P}_k$ by (2). This proves the exact index. When $r = 1$ the index is plainly one.

Finally, a kernel whose coefficient matrix has rank r can be represented using r independent polynomial components, by spectral decomposition. The same argument applies to that minimal support. $\square$

Proof of the bound in Theorem 2. Suppose a kernel in $\mathcal{P}_k$ has p positive and at least one negative eigenvalue. In a minimal decomposition into positive and negative polynomial squares, add back all but one negative square. The resulting kernel stays in $\mathcal{P}_k$ and has signature $(p, 1)$. Its $p + 1$ polynomial components are independent. If $p < 2k$, Theorem 1 would make it positive semidefinite, a contradiction. $\square$

4 Explicit Hankel certificates

Proposition 5. *For each $k \geq 1$, the polynomial*

$$Q_k(z, \bar{z}) = k \sum_{\substack{0 \leq j \leq 2k \\ j \neq k}} |z|^{2j} - |z|^{2k}$$

has signature $(2k, 1)$, belongs to $\mathcal{P}_k$, and does not belong to $\mathcal{P}_{k+1}$. For $k \geq 2$, its square is a holomorphic squared norm.

Proof. Given k complex points z_i and weights a_i , put $x_j = \sum_{i=1}^k a_i z_i^j$. The Hankel matrix

$$A = (x_{i+j})_{0 \leq i, j \leq k} = \sum_{\ell=1}^k a_\ell (1, z_\ell, \dots, z_\ell^k)^T (1, z_\ell, \dots, z_\ell^k)$$

has rank at most k . No conjugation is used in this rank factorization. Let J be the reversing permutation matrix and write $A = x_k J + E$. The anti-diagonal of E is zero. Choose a unit vector in $\ker A$. As J is unitary,

$$\begin{aligned} |x_k|^2 &\leq \|E\|_F^2 \\ &= \sum_{\substack{0 \leq j \leq 2k \\ j \neq k}} \min(j+1, 2k+1-j) |x_j|^2 \leq k \sum_{j \neq k} |x_j|^2. \end{aligned} \quad (4)$$

This proves $Q_k \in \mathcal{P}_k$.

Take the $k+1$ points to be all $(k+1)$ -st roots of unity and give a point z_i weight $a_i = z_i^{-k}/(k+1)$. For $0 \leq j \leq 2k$ the resulting moment x_j is zero except for $x_k = 1$. The quadratic form in (1) is then -1 , proving failure of $\mathcal{P}_{k+1}$ and sharpness in Theorem 2.

For the square, set

$$q_k(x) = k \sum_{j=0}^{2k} x^j - (k+1)x^k.$$

For $0 \leq j \leq 4k$, direct multiplication gives

$$\begin{aligned} [x^j]q_k(x)^2 &= k^2 \min(j+1, 4k+1-j, 2k+1) \\ &\quad - 2k(k+1)\mathbf{1}_{\{k \leq j \leq 3k\}} + (k+1)^2 \mathbf{1}_{\{j=2k\}}. \end{aligned} \quad (5)$$

Outside the indicated band the coefficient is positive. Inside it, dropping the final nonnegative term leaves at least $k(k+1)(k-2) \geq 0$ when $k \geq 2$. Thus every coefficient is nonnegative. Substituting $x = |z|^2$ supplies the required squared norm. $\square$

Indefinite Hermitian polynomials with squared-norm powers are already well established; see [2, 3, 4]. For example, D'Angelo [4, Example 4.1] gives $(1 + |z|^2)^4 - 7|z|^4$ with a squared-norm square. Proposition 5 provides a moment certificate for every positivity level; it is not a claim to have first discovered the qualitative power phenomenon.

5 A complete self-power criterion at rank four

We now apply the rank bound to a real-valued polynomial $R(z, \bar{z})$ with its unique Hermitian polarization. The exponent in the self-power question is always a positive integer.

Lemma 6. *If $R \not\equiv 0$ and $R^N = \sum_j |p_j|^2$ for holomorphic polynomials p_j , then R has constant sign on its nonzero set. For that sign $\sigma \in \{-1, 1\}$, the kernel σR belongs to $\mathcal{P}_2$.*

Proof. Take two points where R is nonzero and restrict to the complex affine line through them. At least one restricted p_j is a nonzero one-variable polynomial. Their common zero set on the line is therefore finite. Its complement is connected, and the continuous real-valued function R has no zero there. Its two specified values have the same sign. Since the points were arbitrary, the sign is constant globally.

Put $S = \sigma R \geq 0$. When $\sigma = -1$, the successful exponent N must be even. In either case S^N is the given squared norm. Polarization and ordinary Cauchy–Schwarz yield

$$|S(z, \bar{w})|^{2N} \leq S(z, \bar{z})^N S(w, \bar{w})^N.$$

Taking positive N -th roots proves the two-point inequality; together with nonnegative diagonal values it gives $S \in \mathcal{P}_2$. This last implication is the established root criterion [3, Lemma 7.1 and Remark 7.1]. $\square$

Theorem 7. *Let $R \not\equiv 0$ have coefficient rank at most four. Some positive power of R is a holomorphic squared norm if and only if its coefficient matrix is positive or negative semidefinite. In the positive case the least successful exponent is one; in the negative case it is two.*

Proof. If a power succeeds, Lemma 6 and Theorem 1, with $k = 2$, show that σR is itself a squared norm. Thus the coefficient matrix of R has the asserted sign. Conversely, a positive semidefinite coefficient matrix factors as a Gram matrix. If instead $-R = \sum_j |h_j|^2$, then

$$R^2 = \sum_{i,j} |h_i h_j|^2.$$

A nonzero negative R cannot itself, or in any odd power, be a squared norm. $\square$

Corollary 8. *Theorem 7 applies to every one-variable Hermitian polynomial of bidegree at most $(3, 3)$. Both its rank and degree cutoffs are sharp as universal statements.*

Proof. The monomial coefficient matrix has size at most four. For sharpness, the polynomial from Proposition 5 at $k = 2$ has coefficient vector $(2, 2, -1, 2, 2)$ and

$$(2 + 2x - x^2 + 2x^3 + 2x^4)^2 = 4 + 8x + 4x^3 + 17x^4 + 4x^5 + 8x^7 + 4x^8.$$

It has rank five, bidegree $(4, 4)$ and an indefinite coefficient matrix, while its square is a squared norm. $\square$

6 Scope and supplementary verification

The AIM question asks for a criterion without rank or degree restrictions [1]; that problem is not resolved here. The result is a complete low-rank criterion and a sharp stability theorem for polynomial kernels. The zero polynomial is a trivial squared norm if zero summands are allowed.

The global polynomial domain matters. On just three points, the matrix with diagonal entries one and off-diagonal entries $-3/4$ is indefinite, although its two-point restrictions and its entrywise square are positive definite. Its determinant is $-49/32$, and the determinant of its entrywise square is $833/2048$. Interpolating the three evaluation vectors by polynomials does not preserve positivity away from those points.

The accompanying standard-library Python program checks 22,703 finite identities and inequalities: confluent determinants, sparse and mixed polynomial Jacobians, exact Gaussian-integer moments, rational Hankel determinants, cyclotomic moment filters, coefficient formulas, and squared-factor cancellation congruences. Normal and optimized executions agree. The program also rejects the finite-domain extension, a leading-principal-minors-only test, and the false $k = 1$ square extension. These computations supplement the written proof, not its universal quantifiers.

This unrefereed preprint was developed with AI assistance in reasoning, literature searches, symbolic checks and manuscript preparation. It is self-audited, not independently reviewed or formally verified; absolute priority for the stated combination of classical methods is not certified.

References

- [1] American Institute of Mathematics, *Emerging applications of complexity for CR mappings*, workshop problems, 2010, p. 1, Squared norms Problem 2. <https://aimath.org/WWN/crmappings/crmappings.pdf>.
- [2] J. P. D’Angelo and D. Varolin, Positivity conditions for Hermitian symmetric functions, *Asian J. Math.* **7** (2003), 1–18. [arXiv:math/0306220](https://arxiv.org/abs/math/0306220).
- [3] J. P. D’Angelo, Hermitian analogues of Hilbert’s 17-th problem, *Adv. Math.* **226** (2011), 4607–4637. [doi:10.1016/j.aim.2010.12.013](https://doi.org/10.1016/j.aim.2010.12.013). See also the published corrigendum.
- [4] J. P. D’Angelo, Positivity Conditions for Hermitian Polynomials, *J. Geom. Anal.* **35** (2025), article 387. [doi:10.1007/s12220-025-02228-5](https://doi.org/10.1007/s12220-025-02228-5).
- [5] The Stacks Project Authors, Chevalley’s theorem, Tag 054K. <https://stacks.math.columbia.edu/tag/054K>.