

Infinite Boundary-Jet Ambiguity for Proper Embeddings into a Smooth Strongly Convex Domain

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Abstract

For every $n \geq 2$, we construct one bounded domain $D \subset \mathbb{C}^{n+1}$ with smooth strongly convex boundary and a circle of distinct proper holomorphic embeddings $\mathbb{B}^n \rightarrow D$. All embeddings extend smoothly to the closed ball and have the same infinite jet at one fixed boundary point. The domain can be chosen arbitrarily close to the unit ball in any prescribed finite smoothness norm on its defining function. A holomorphic function flat at the boundary point supplies the graphs, and a compactly supported perturbation of the quadratic defining function places all graphs in the same target. For comparison, an elementary orthogonality argument shows that a proper rational ball map of degree at most d is determined by its d -jet at the interior origin, even among all holomorphic maps to the target ball. The boundary example concerns a prescribed boundary point; it does not resolve the general optimal dimension-dependent interior jet order.

1 The distinction between interior and boundary jets

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$ and $S^{2n-1} = \partial\mathbb{B}^n$. At an interior point, $j_p^k F$ denotes the holomorphic Taylor polynomial of order k . For a map smooth on a closed domain, equality of boundary jets means equality of all derivatives up to that order of smooth extensions at the boundary point. These jets are independent of the chosen extensions, since the extensions agree on the open domain and their derivatives are continuous. Equality of infinite jets means equality for every finite order.

Problem 1 of the AIM workshop list on CR mappings [1] asks two related questions. Its first part asks for the least dimension-dependent order determining proper rational maps between balls at the origin. Its second part asks about finite determination for proper holomorphic maps between strongly pseudoconvex domains that extend smoothly to the boundary. The second part says *at a point* without specifying its location. We keep the two locations separate.

Finite determination for CR embeddings is known under additional hypotheses. Ebenfelt and Lamel [3] obtain results for smooth embeddings of constant degeneracy near the base point. The survey [4] explains the distinction between these results and the later theory for Nash targets; see also [5]. The smooth target below is not asserted to be real-analytic or Nash. Transversality by itself is not the constant-degeneracy hypothesis.

Our principal assertion is the following explicit boundary nonuniqueness result.

Theorem 1. *For every $n \geq 2$, there are a bounded domain $D \subset \mathbb{C}^{n+1}$ with C^∞ strongly convex boundary, a point $p \in S^{2n-1}$, and a family*

$$F_c : \mathbb{B}^n \longrightarrow D, \quad c \in \Gamma := \{c \in \mathbb{C} : |c|^2 = \operatorname{Re} c\},$$

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of distinct proper holomorphic embeddings, smooth on $\overline{\mathbb{B}^n}$, such that

$$j_p^k F_c = j_p^k F_b \quad \text{for all } b, c \in \Gamma \text{ and all } k \geq 0.$$

The source, target, and boundary point are fixed for the whole family. Each embedding is transversal along the boundary. For any integer $m \geq 0$ and any $\delta > 0$, the domain may be chosen with a global defining function ρ satisfying

$$\|\rho - (\|z\|^2 + |u|^2 - 1)\|_{C^m(\mathbb{C}^{n+1})} < \delta.$$

The norm in the last assertion is applied to the compactly supported difference of the defining functions. In particular, even an infinite jet at a specified boundary point does not determine a proper embedding in the smooth strongly convex setting.

2 A smooth holomorphic function flat at the boundary

Fix $p = (1, 0, \dots, 0) \in S^{2n-1}$. On $\mathbb{B}^n$ define

$$h(z) = \exp(-(1 - z_1)^{-1/2}), \tag{1}$$

where the half-power uses the branch on the right half-plane. This function is holomorphic and nonzero in the ball. We need a smooth ambient extension, rather than only a radial estimate.

Lemma 2. *There is $H \in C_c^\infty(\mathbb{C}^n)$ such that $H = h$ on $\mathbb{B}^n$, all real partial derivatives of H vanish at p , and $H(0) = e^{-1}$.*

Proof. Write $w = 1 - z_1$. For $z \in \overline{\mathbb{B}^n} \setminus \{p\}$ one has $|\arg w| \leq \pi/2$. On the wider sector $|\arg w| \leq 3\pi/4$, the principal branch satisfies

$$\operatorname{Re}(w^{-1/2}) \geq \cos(3\pi/8) |w|^{-1/2}. \tag{2}$$

Every complex derivative of $\exp(-w^{-1/2})$ is this exponential times a polynomial in $w^{-1/2}$. Thus every such derivative tends to zero faster than any power of $|w|$, uniformly on the indicated sector.

Choose a smooth angular cutoff η equal to one for $|\theta| \leq \pi/2$ and supported in $|\theta| < 3\pi/4$. For $w \neq 0$, set

$$E(w) = \eta(\arg w) \exp(-w^{-1/2}),$$

using the principal branch on the support of η , and set $E = 0$ near the negative real axis. Set $E(0) = 0$. The definition is smooth away from zero. Each derivative of the angular factor costs at most an inverse power of $|w|$; estimate (2) dominates all these powers. Consequently every real derivative extends by zero at zero, and E is smooth and flat there.

Choose $\beta \in C_c^\infty(\mathbb{C}^n)$ equal to one on $\{\|z\|^2 \leq 2\}$ and zero on $\{\|z\|^2 \geq 3\}$. Then

$$H(z) = \beta(z)E(1 - z_1)$$

has the required properties. In particular, at the origin $w = 1$, so $H(0) = e^{-1}$. □

3 One strongly convex target for all graphs

Choose a real smooth function χ on $\mathbb{C}$, equal to one on $\{|u| \leq 1\}$ and zero on $\{|u| \geq 2\}$, and define

$$\Phi(z, u) = \chi(u) \operatorname{Re}(u \overline{H(z)}). \quad (3)$$

This is a real-valued compactly supported smooth function. Put

$$M_0 = \sup_{\mathbb{C}^n} |H|, \quad M_2 = \sup_{\mathbb{C}^{n+1}} \|D^2\Phi\|_{\text{op}},$$

where the Hessian is taken in real coordinates. Both numbers are finite. Choose

$$0 < \varepsilon < \min\{(1 + M_0)^{-1}, (1 + M_2)^{-1}\}. \quad (4)$$

Set

$$\rho(z, u) = \|z\|^2 + |u|^2 - 1 - \varepsilon\Phi(z, u), \quad D = \{(z, u) : \rho(z, u) < 0\}. \quad (5)$$

Lemma 3. *The set D is a bounded domain with smooth strongly convex boundary. In particular, it is strongly pseudoconvex.*

Proof. By (4),

$$D^2\rho \succeq (2 - \varepsilon M_2)\text{Id} \succ \text{Id}$$

on the entire ambient space. Also, ρ is coercive, because its perturbation of the quadratic function has compact support, and $\rho(0, 0) = -1$. Its negative sublevel set is therefore nonempty, bounded, and convex, hence connected. Every critical point of the strictly convex function ρ is its global minimizer, with value at most -1 . Thus zero is a regular value. The real Hessian is positive on every tangent space to $\{\rho = 0\}$, which proves strong convexity. Positivity of the real Hessian also implies positivity of the Levi form. $\square$

The cutoff χ is important: without it, the second z -derivatives of $\operatorname{Re}(u \overline{H(z)})$ need not remain bounded as $|u|$ tends to infinity. The estimate above is global and proves boundedness and convexity of the actual target, not merely of a local hypersurface germ.

Proof of Theorem 1. The set Γ is the circle

$$\left|c - \frac{1}{2}\right| = \frac{1}{2};$$

in particular, $|c| \leq 1$. Define

$$F_c(z) = (z, c\varepsilon h(z)). \quad (6)$$

By (4), $|c\varepsilon h(z)| < 1$ on the closed ball, so $\chi = 1$ along every graph. Since $H = h$ there, direct substitution gives the exact pullback identity

$$\begin{aligned} \rho(F_c(z)) &= \|z\|^2 - 1 + \varepsilon^2(|c|^2 - \operatorname{Re} c)|h(z)|^2 \\ &= \|z\|^2 - 1. \end{aligned} \quad (7)$$

Hence F_c maps the ball into D and its sphere into ∂D .

If $K \subset D$ is compact, then $\rho \leq -a$ on K for some $a > 0$. Identity (7) confines $F_c^{-1}(K)$ to $\{\|z\|^2 \leq 1 - a\}$; the preimage is closed in the ball, so it is compact or empty. This proves properness. The first n coordinates of F_c are z , so it is an injective immersion with continuous inverse given by

the coordinate projection. It is a holomorphic embedding. Lemma 2 gives its smooth extension to the closed ball.

For $b, c \in \Gamma$,

$$F_c - F_b = (0, (c - b)\varepsilon h).$$

All derivatives of this difference vanish at p . But $F_c(0) = (0, c\varepsilon e^{-1})$, so different parameters give different maps. All boundary values at p equal $(p, 0)$. Differentiating (7) on the closed ball shows that its normal derivative on the sphere is nonzero, proving transversality.

Finally, for fixed m ,

$$\|\rho - (\|z\|^2 + |u|^2 - 1)\|_{C^m} = \varepsilon \|\Phi\|_{C^m}.$$

Decreasing the positive number ε proves the last assertion while keeping all maps distinct. For each chosen ε , the domain is fixed for every parameter and every jet order. $\square$

4 Interior determination for rational ball maps

The boundary phenomenon contrasts with a direct interior uniqueness argument.

Proposition 4. *Let $F : \mathbb{B}^n \rightarrow \mathbb{B}^N$ be a proper holomorphic rational map with a representation $F = P/q$, where P is a vector polynomial, $q \not\equiv 0$ is a scalar polynomial, and every component of P and q has degree at most d . If $G : \mathbb{B}^n \rightarrow \mathbb{B}^N$ is holomorphic and $j_0^d G = j_0^d F$, then $G = F$. No rationality or properness assumption on G is required.*

Proof. Let $V = qG - P = q(G - F)$. Its Taylor expansion at zero has no homogeneous terms of degree at most d . Integrate over the unit sphere with normalized surface measure σ . Homogeneous holomorphic polynomials of different total degrees are orthogonal: rotation $z \mapsto e^{it}z$ multiplies the inner product of terms of degrees a, b by $e^{i(a-b)t}$. The Taylor series of V converges uniformly on rS^{2n-1} for $0 < r < 1$. Since $\deg P \leq d$, it follows that

$$\int \|q(r\zeta)G(r\zeta)\|^2 d\sigma = \int \|P(r\zeta)\|^2 d\sigma + \int \|V(r\zeta)\|^2 d\sigma.$$

Using $\|G\| \leq 1$, we obtain

$$0 \leq \int \|V(r\zeta)\|^2 d\sigma \leq \int (|q(r\zeta)|^2 - \|P(r\zeta)\|^2) d\sigma. \quad (8)$$

Properness implies $\|F(z)\| \rightarrow 1$ as $\|z\| \rightarrow 1$: otherwise a sequence escaping all compact subsets of the source would have images in a compact subset of the target. Thus $\|P\|^2 = |q|^2$ on sphere points where $q \neq 0$. These points are dense. Indeed, the restriction of a nonzero holomorphic polynomial to the sphere cannot vanish on an open patch: real-analytic continuation on the sphere, followed by the maximum principle in the ball, would make the polynomial zero. By continuity the identity holds on the whole sphere. Therefore the right side of (8) tends to zero as $r \uparrow 1$.

If V_ℓ were a nonzero homogeneous Taylor term, the left side of (8) would be at least

$$r^{2\ell} \int \|V_\ell(\zeta)\|^2 d\sigma,$$

whose limit is positive. Hence $V = 0$. On the nonempty open set $q \neq 0$ we have $G = F$, and the identity theorem gives equality throughout the ball. $\square$

This proof also permits a representation with $q(0) = 0$; only the holomorphic identity $qF = P$ is needed. The argument is elementary Hardy-space orthogonality. The related polynomial statement up to norm equivalence in [2, p. 16 of the arXiv version] is different from literal equality of maps in fixed coordinates.

For $N \geq n \geq 2$, let $k(n, N)$ be the least order determining all proper rational maps $\mathbb{B}^n \rightarrow \mathbb{B}^N$ at zero, in fixed coordinates. The degree estimate of D'Angelo and Lebl [2, Corollary 2] gives

$$\left\lfloor \frac{N-1}{n-1} \right\rfloor \leq k(n, N) \leq \left\lfloor \frac{N(N-1)}{2(2n-3)} \right\rfloor. \quad (9)$$

For completeness, the lower obstruction follows from the standard iterated Whitney maps

$$W_d(z', w) = (z', wz', \dots, w^{d-1}z', w^d) \in \mathbb{C}^{d(n-1)+1}.$$

They satisfy

$$1 - \|W_d(z', w)\|^2 = (1 - \|z'\|^2 - |w|^2)(1 + |w|^2 + \dots + |w|^{2d-2}).$$

Rotating the last coordinate produces distinct proper maps with the same $(d-1)$ -jet. Appending zero coordinates proves the lower bound in (9).

For example, the upper bound gives $k(2, 3) \leq 3$. The classical cubic sphere map, with a target phase,

$$C_\theta(z, w) = (e^{i\theta}z^3, \sqrt{3}zw, w^3),$$

gives the reverse bound. Indeed, for $x = |z|^2$, $y = |w|^2$ and $s = x + y$,

$$1 - x^3 - 3xy - y^3 = (1 - s)(s^2 + s + 1 - 3xy),$$

and the second factor is $x^2 - xy + y^2 + x + y + 1 > 0$. Thus these are proper ball maps, and different phases give the same 2-jet. Consequently $k(2, 3) = 3$. The degree bound and the classical Whitney and cubic families are prior constructions; this corollary is not presented as a new classification.

5 Scope

Theorem 1 rules out finite or infinite jet determination at an arbitrary prescribed boundary point for general smooth strongly pseudoconvex targets, even for transversal embeddings in complex codimension one. It does not rule out determination at another point: the displayed family is already separated by its value at the interior origin. The construction does not provide a real-analytic or Nash target and does not contradict the corresponding positive results. Nor is it a statement about self-maps or CR diffeomorphisms in equal ambient dimensions.

Proposition 4 and (9) do not determine the optimal order for every pair (n, N) . Accordingly, the two-part AIM question is not claimed resolved in full. The contribution of the boundary construction is its explicit global realization in a single bounded strongly convex target. No absolute priority claim is made.

Exact rational checks of the circle parametrization, the cubic and Whitney identities, and derivatives of the flat function accompany the source package. They are consistency checks; smooth extension, global convexity, properness, and uniqueness are established by the analytic proofs above. This is an unrefereed preprint. AI assistance was used in developing the arguments, checking calculations, searching sources, and preparing the manuscript; the author is responsible for its contents.

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