

Lower Bounds for Positive-Power Exponents in a Hermitian Polynomial Family

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Abstract

For a fixed integer $k \geq 2$, let $N_k(t)$ be the least positive integer n for which $((1+x)^{2k} - tx^k)^n$ has nonnegative coefficients. Known positivity theorems imply that $N_k(t)$ is finite for $0 \leq t < T_k = 2^{2k-1}$, whereas no exponent works at $t = T_k$. We give an explicit necessary lower bound as t approaches this boundary. Writing $\epsilon = T_k - t$, $L = \log(1/\epsilon)$ and $\beta_k = T_k(k-1)/(4k)$, we prove $\liminf_{\epsilon \downarrow 0} (\epsilon N_k(T_k - \epsilon) - \beta_k L - \beta_k \log L) \geq C_k$, with an explicit Gamma-function constant C_k . The proof compares a Gaussian contribution to a selected coefficient with a negative contribution flat to order $2k$. The estimates are uniform in t near the boundary. Equivalently, they bound the exponent needed to express a power of a known bihomogeneous Hermitian form as a holomorphic squared norm. No matching upper bound or general criterion for powers of Hermitian polynomials is asserted.

1 The family and the quantitative statement

For an integer $k \geq 2$ and a real parameter t , put

$$p_{k,t}(x) = (1+x)^{2k} - tx^k, \quad (1)$$

$$R_{k,t}(z, w) = (|z|^2 + |w|^2)^{2k} - t|zw|^{2k}. \quad (2)$$

The family (2) is due to the established study of positivity conditions for Hermitian forms; see D’Angelo–Varolin [6] and D’Angelo [5]. Set

$$T = T_k = 2^{2k-1}, \quad \gamma_k = \frac{k-1}{2k}, \quad \beta_k = \frac{T(k-1)}{4k}. \quad (3)$$

For $0 \leq t < T$, define

$$N_k(t) = \min\{n \geq 1 : p_{k,t}(x)^n \text{ has only nonnegative coefficients}\}.$$

Existence of this minimum follows from known theorems, as recalled in Section 2. Since (2) is diagonal in the monomial basis, this is also the least n for which $R_{k,t}^n$ is a sum of squared moduli of holomorphic polynomials.

Define the positive constants

$$A_* = (2\pi k)^{-1/2}, \quad B_* = \frac{\Gamma(1/(2k))T^{1/(2k)}}{2\pi k}, \quad C_k = \beta_k \log \beta_k + \frac{T}{2} \log \frac{B_*}{A_*}. \quad (4)$$

All logarithms in this note are natural.

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Theorem 1. Fix $k \geq 2$. With $\epsilon = T - t$ and $L = \log(1/\epsilon)$,

$$\liminf_{\epsilon \downarrow 0} (\epsilon N_k(T - \epsilon) - \beta_k L - \beta_k \log L) \geq C_k. \quad (5)$$

In particular,

$$\liminf_{\epsilon \downarrow 0} \frac{\epsilon N_k(T - \epsilon)}{\log(1/\epsilon)} \geq \beta_k. \quad (6)$$

The same necessary estimates hold for any choice of successful exponents, not only the least one.

For the quartic polynomial, $T = 8$ and $\beta_2 = 1$, so

$$\liminf_{\epsilon \downarrow 0} (\epsilon N_2(8 - \epsilon) - L - \log L) \geq 4 \log \left(\frac{\Gamma(1/4) 8^{1/4}}{2\sqrt{\pi}} \right) = 2.16948 \dots \quad (7)$$

The estimates are for fixed k ; their constants are not uniform as $k \rightarrow \infty$. They are lower bounds, not an asymptotic equivalence.

Relation to previous work

De Angelis developed coefficient asymptotics for high powers and proved eventual positivity under strong positivity and suitable endpoint conditions [3]. His earlier work includes reciprocal quartics with a negative middle coefficient [2], and his later work treats regimes in which contributions from several maxima matter [4]. The Fourier and Laplace methods used below are standard instances of this approach, not a new general asymptotic method.

D’Angelo–Varolin [6, Theorem 3] determine the two-point positivity threshold T for (2). D’Angelo explicitly discusses the need for higher powers as the quartic parameter approaches 8 [5, Example 4.1], and the corresponding sextic example with threshold 32 [5, Example 4.2]. Our purpose is to calculate the necessary rate in (5), by retaining the competing degenerate contribution as the parameter approaches the boundary.

Michelen–Sahasrabudhe [7, Theorem 3] characterize eventual nonnegativity of real univariate polynomials, giving a known criterion for binary diagonal Hermitian forms. Neither that classification nor the qualitative thresholds of the present family are claimed here as new results. The general problem of characterizing bihomogeneous Hermitian forms with a holomorphic squared-norm power, posed in the AIM workshop [1, Mappings, Problem 3.2], is not resolved by this family calculation.

2 Existence below the boundary

A diagonal bihomogeneous Hermitian polynomial is a squared norm of a holomorphic polynomial mapping exactly when every coefficient is nonnegative. Necessity follows because its unique monomial coefficient matrix must be positive semidefinite. For sufficiency, use the distinct monomials multiplied by the square roots of their positive coefficients; these holomorphic polynomials are linearly independent. Allowing nonhomogeneous summands does not change necessity: diagonal coefficients outside the required degree are sums of squared moduli and force the corresponding coefficients of every summand to vanish.

For $u, v \in \mathbb{C}^2$, polarize (2) as

$$K_t(u, v) = \langle u, v \rangle^{2k} - t(u_1 u_2 \overline{v_1 v_2})^k.$$

By [6, Theorem 3(2)], K_T belongs to $\mathcal{P}_2$: each two-point evaluation matrix is positive semidefinite. For $0 \leq t < T$, write

$$K_t = (1 - t/T)K_0 + (t/T)K_T.$$

For two nonzero vectors not proportional over $\mathbb{C}$, the matrix for K_0 is positive definite by strict Cauchy–Schwarz. Its positive-weight sum with the positive semidefinite matrix for K_T remains positive definite. Apply this to

$$u = (\sqrt{r}e^{i\theta}, 1), \quad v = (\sqrt{r}, 1), \quad r > 0, \quad \theta \notin 2\pi\mathbb{Z}.$$

The diagonal entries are $p_{k,t}(r) > 0$, and the off-diagonal entry is $p_{k,t}(re^{i\theta})$. Therefore

$$|p_{k,t}(z)| < p_{k,t}(|z|) \quad (z \notin \mathbb{R}_{\geq 0}). \quad (8)$$

The constant, linear, next-to-leading and leading coefficients of $p_{k,t}$ are $1, 2k, 2k, 1$, since $k \geq 2$. De Angelis’s eventual-positivity theorem [3], also stated in [7, Theorem 1], now implies that every sufficiently large power has strictly positive coefficients. This proves that $N_k(t)$ is finite, using known results.

At $t = T$, however,

$$p_{k,T}(1) = T, \quad p_{k,T}(-1) = (-1)^{k+1}T.$$

If $p_{k,T}^n$ had nonnegative coefficients, equality in $|p_{k,T}(-1)^n| \leq p_{k,T}(1)^n$ would force all its nonzero coefficients to have the same parity. Its constant coefficient is 1 and its linear coefficient is $2kn > 0$, a contradiction. Thus no exponent succeeds at the boundary. In particular,

$$N_k(t) \longrightarrow \infty \quad (t \uparrow T). \quad (9)$$

Indeed, a bounded sequence of successful exponents would have a constant subsequence; coefficient-wise continuity would then give a successful exponent at T .

3 A uniform coefficient estimate

For the rest of the proof restrict t to the compact interval $[3T/4, T]$, and put

$$a = 2T - t, \quad \alpha = kT/2, \quad A(t) = \frac{\sqrt{a/\alpha}}{2\sqrt{\pi}}, \quad B(t) = \frac{\Gamma(1/(2k))t^{1/(2k)}}{2\pi k}. \quad (10)$$

Let $\ell_n = 0$ for odd n and $\ell_n = 1$ for even n . In particular, $n + \ell_n$ is always odd.

Lemma 2. *Uniformly for $t \in [3T/4, T]$, as $n \rightarrow \infty$,*

$$\begin{aligned} [x^{kn+\ell_n}]p_{k,t}(x)^n &= A(t)a^n n^{-1/2}(1 + O_k(n^{-1})) \\ &\quad - B(t)t^n n^{-1/(2k)}(1 + O_k(n^{-1/k})). \end{aligned} \quad (11)$$

The two errors are relative to their respective summands, not to their potentially small difference.

Proof. On the unit circle the phase-normalized polynomial is real:

$$q_t(\theta) := e^{-ik\theta}p_{k,t}(e^{i\theta}) = (2\cos(\theta/2))^{2k} - t.$$

Fourier coefficient extraction and evenness give, for $\ell \in \{0, 1\}$,

$$[x^{kn+\ell}]p_{k,t}^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} q_t(\theta)^n \cos(\ell\theta) d\theta. \quad (12)$$

On $[0, \pi]$, q_t decreases strictly from a to $-t$. Its zero stays a positive distance from both endpoints, uniformly in t . Choose small fixed neighborhoods of 0 and π , with the latter viewed across the ends of the circle. Outside them, the positive part of q_t is at most $a(1 - \eta)$ and the absolute value of its negative part is at most $t(1 - \eta)$, for one uniform $\eta > 0$. The middle contributions are exponentially small relative to their respective endpoint terms.

Near zero, uniformly in t ,

$$\begin{aligned} q_t(\theta) &= a - \alpha\theta^2 + O_k(\theta^4), \\ \log(q_t(\theta)/a) &= -(\alpha/a)\theta^2 + O_k(\theta^4). \end{aligned}$$

Here a has positive upper and lower bounds. On a sufficiently small fixed neighborhood the logarithm is at most $-c\theta^2$. To justify the error estimate explicitly, compare the two exponentials using the mean value inequality. The difference is bounded by $Cn\theta^4 e^{-c'n\theta^2}$, whose integral is $O_k(n^{-3/2})$. The amplitude $\cos(\ell\theta) = 1 + O(\theta^2)$ has an error of the same integrated order, uniformly for both choices of ℓ . The Gaussian integral, extended to the real line with exponentially small error, therefore contributes

$$\frac{a^n}{2\pi} \sqrt{\frac{a}{n\alpha}} \sqrt{\pi} (1 + O_k(n^{-1})) = A(t) a^n n^{-1/2} (1 + O_k(n^{-1})). \quad (13)$$

Write $u = \theta - \pi$ at the other endpoint, using periodic continuation. There,

$$\begin{aligned} q_t(\pi + u) &= -t + u^{2k} + O_k(u^{2k+2}), \\ \log(|q_t(\pi + u)|/t) &= -u^{2k}/t + O_k(u^{2k+2} + u^{4k}). \end{aligned}$$

The logarithm is uniformly at most $-c|u|^{2k}$. Since $k \geq 2$, the remainder on a fixed small neighborhood is $O_k(u^{2k+2})$. The same exponential comparison now bounds its integrated effect by

$$Cn \int_{\mathbb{R}} |u|^{2k+2} e^{-c'u|^{2k}} du = O_k(n^{-1/(2k)-1/k}).$$

The amplitude is $\cos(\ell(\pi + u)) = (-1)^\ell (1 + O(u^2))$ and has the same relative error order. Counting the two halves of the endpoint on the circle once, the model integral is

$$\int_{\mathbb{R}} e^{-nu^{2k}/t} du = \frac{\Gamma(1/(2k))}{k} (t/n)^{1/(2k)}.$$

Its sign in (12) is $(-1)^{n+\ell}$, which is negative when $\ell = \ell_n$. Together with (13), this proves (11), including uniformity as $t \uparrow T$. $\square$

4 Proof of the lower bound

If $p_{k,t}^n$ has nonnegative coefficients, the coefficient in Lemma 2 is nonnegative. For sufficiently large n the two separate error factors are positive, so taking logarithms gives

$$n \log \frac{T + \epsilon}{T - \epsilon} \geq \gamma_k \log n + \log \frac{B(t)}{A(t)} - Cn^{-1/k}, \quad (14)$$

for a constant C depending only on k . Here and below one may restrict to $\epsilon > 0$ sufficiently small.

Take $n = N_k(T - \epsilon)$. By (9), $n \rightarrow \infty$. Along any sequence with $n \geq \epsilon^{-2}$, the expression inside the liminf in (5) tends to $+\infty$. On the remaining sequences $n < \epsilon^{-2}$, and

$$n \log \frac{T + \epsilon}{T - \epsilon} = \frac{2\epsilon n}{T} + O_k(n\epsilon^3) = \frac{2\epsilon n}{T} + o(1).$$

Since $A(t) \rightarrow A_*$ and $B(t) \rightarrow B_*$, put $x = \epsilon n$ and $C_0 = (T/2) \log(B_*/A_*)$. Inequality (14) yields

$$x \geq \beta_k \log n + C_0 + o(1). \quad (15)$$

This first implies $x \rightarrow \infty$, because $n \rightarrow \infty$ and $\beta_k > 0$. Eventually $x \geq 1$, and $\log n = L + \log x$, so

$$x \geq \beta_k L + C_0 + o(1), \quad \log x \geq \log L + \log \beta_k + o(1).$$

Substitution in (15) proves

$$x \geq \beta_k L + \beta_k \log L + \beta_k \log \beta_k + C_0 + o(1),$$

which is (5). The proof uses only that n is a successful exponent; any sequence of such exponents must diverge by the same boundary argument. This proves the final assertion of Theorem 1 as well.

5 Exact examples and limitations

The supplementary integer-arithmetic checker gives the following least exponents. Each earlier power has a retained negative coefficient; every coefficient at the first success was independently checked by the binomial expansion of (1).

t	7	15/2	31/4	63/8	127/16	t	255/32
$N_2(t)$	2	8	20	48	108	$N_2(t)$	242

Normal and optimized Python runs agree. A separate 70-digit numerical diagnostic compares the Fourier integral with exact coefficient expressions in 27 cases, for $k = 2, 3, 4$. It tests the two prefactors and phase choice; it is not interval-certified integration. These finite checks do not replace the uniform proof.

The sign of a selected coefficient is only a necessary condition for positivity of all coefficients. Thus this argument supplies neither a matching upper bound for $N_k(t)$ nor a parity law for successful exponents. A sharp asymptotic would require control over every coefficient index. No conclusion is drawn for arbitrary nondiagonal Hermitian forms.

This AI-assisted preprint is self-audited and unrefereed. It has not received independent review or proof-assistant verification. Its contribution is the explicit calculation above using established positivity and asymptotic methods; the bounded literature search does not certify absolute priority.

References

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