An Explicit Rational Homotopy from the Faran Map to a Linear Map in Target Dimension Four

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Abstract

We construct a homotopy through proper rational holomorphic maps from the unit ball in $\mathbb{C}^2$ to the unit ball in $\mathbb{C}^4$, joining $(z,w) \mapsto (z^3, \sqrt{3}zw, w^3, 0)$ to $(z,w) \mapsto (z,w,0,0)$. The construction answers the explicit target-four question in the AIM list on the Cauchy–Riemann equations. A mixed-term polynomial family of coefficient rank four joins a unitary transform of the Faran map to a partially tensored quadratic map. An explicit rational segment then degenerates to a Whitney map, which is joined to the linear embedding. The full family is jointly continuous on the closed source ball, and each slice extends holomorphically past that ball. All slices have rational degree at most three. We prove a uniform estimate at the degenerating endpoint and explain why this particular path is not continuous in the C^1 boundary topology. No classification of arbitrary ball-map homotopies is asserted.

1 The question and the result

Write $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$. The concrete special case in Problem 3.1 of the AIM list *The Cauchy–Riemann equations in several variables*, attributed there to Lebl, asks whether the maps

$$\Phi(z,w) = (z^3, \sqrt{3}zw, w^3, 0), \qquad L(z,w) = (z,w,0,0) \tag{1}$$

are homotopic through proper maps $\mathbb{B}^2 \to \mathbb{B}^4$ [1]. The middle component of Φ has degree two, not three.

We use homotopy through rational maps in the sense of D'Angelo–Lebl [3, Definition 2.2]: the family is continuous, every slice is proper and rational, and the target dimension stays fixed. Their Example 3.1 gives target-four homotopies between the other three Faran representatives and gives a target-five homotopy involving Φ. Their Corollary 3.2 separates the four representatives through rational homotopies in target dimension three. The construction below handles the remaining target-four pair explicitly.

Theorem 1.1. *There is a continuous map $\mathcal{H} : \overline{\mathbb{B}^2} \times [0,1] \to \overline{\mathbb{B}^4}$ such that $\mathcal{H}_0 = \Phi$, $\mathcal{H}_1 = L$, and each slice restricts to a proper rational holomorphic map $\mathbb{B}^2 \to \mathbb{B}^4$ preserving the origin. Every slice extends holomorphically to some neighborhood of $\overline{\mathbb{B}^2}$ and has rational degree at most three.*

The neighborhood may depend on the slice. We do not claim a common extension neighborhood, joint C^1 boundary regularity, or an entirely polynomial path. The theorem answers the displayed special case, not the broader AIM request for general homotopy invariants.

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The principal formula is the polynomial bridge in section 3. Its mixed terms permit a four-dimensional target while its coefficients vary. Rational untensoring and Whitney-type paths are standard tools; automorphism insertion in sphere maps appears, for example, in Reiter [4, Theorem 6]. We prove every identity and endpoint estimate needed for our particular construction directly. No absolute priority claim is made for the construction or its constituent operations.

2 Two elementary facts

Lemma 2.1. *Suppose F is holomorphic on a neighborhood of $\overline{\mathbb{B}^n}$, $F(0) = 0$, and $\|F\| = 1$ on $\partial\mathbb{B}^n$. Then $F : \mathbb{B}^n \to \mathbb{B}^N$ is proper.*

Proof. For each fixed unit vector $e \in \mathbb{C}^N$, the maximum principle applied to $\langle F, e\rangle$ gives $|\langle F, e\rangle| \le 1$ in the ball. Taking the supremum over e gives $\|F\| \le 1$. If equality held at an interior point p, the holomorphic scalar function $\langle F, F(p)\rangle$ would attain its maximum value one there. It would be constant, contradicting $F(0) = 0$. Thus $\|F\| < 1$ in the interior. For a compact set $K \subset \mathbb{B}^N$, the closed preimage in $\overline{\mathbb{B}^n}$ misses the boundary, where $\|F\| = 1$. It is therefore a compact subset of $\mathbb{B}^n$. □

Lemma 2.2. *Unitary changes of source and target coordinates may be traversed by homotopies preserving all the properties in theorem 1.1.*

Proof. For $U \in U(m)$, choose $V \in U(m)$ and real numbers θ_j with $U = V \operatorname{diag}(e^{i\theta_j})V^*$. Then $U_s = V \operatorname{diag}(e^{is\theta_j})V^*$ joins the identity to U. Precomposition and postcomposition by these paths preserve properness, rational degree, and the asserted continuity and extension properties. They also preserve the origin. □

3 A four-coordinate polynomial bridge

For $0 \le g \le 1$, set

$$\begin{aligned} A &= \frac{1-g}{2(2-g)}, & B &= \frac{(3-g)(3-2g)}{2(2-g)}, \\ C &= \frac{(3-2g)(1-g)}{2(2-g)}, & D &= \frac{3-g}{2(2-g)}. \end{aligned} \tag{2}$$

These coefficients are nonnegative and their common denominator is at least two. Square roots below always mean nonnegative real roots. Define

$$P_g(u, v) = \begin{pmatrix} \sqrt{g}\, u \\ \sqrt{D}\, v^2 - \sqrt{C}\, u^2 \\ \sqrt{A}\, u^3 + \sqrt{B}\, uv^2 \\ \sqrt{B}\, u^2 v + \sqrt{A}\, v^3 \end{pmatrix}. \tag{3}$$

Proposition 3.1. *Each P_g is a proper polynomial map $\mathbb{B}^2 \to \mathbb{B}^4$. The family is jointly continuous on $\overline{\mathbb{B}^2} \times [0, 1]$. Its initial point is unitarily equivalent to Φ, and its final point is*

$$E(u, v) = (u, v^2, uv^2, u^2 v). \tag{4}$$

Proof. Put $x = |u|^2$, $y = |v|^2$, and $r = \operatorname{Re}(u^2\overline{v}^2)$. Expansion of the four components gives

$$\begin{aligned} \|P_g\|^2 = {} & A(x^3 + y^3) + B(x^2 y + xy^2) + Cx^2 + Dy^2 + gx \\ & + 2r\big(\sqrt{AB}(x + y) - \sqrt{CD}\big). \end{aligned} \tag{5}$$

The following are rational identities in g:

$$\begin{gathered} AB = CD, \qquad A + C + g = 1, \qquad B + C + 2g = 3, \\ B + D + g = 3, \qquad A + D = 1. \end{gathered} \tag{6}$$

Since the roots are nonnegative, the mixed term in (5) vanishes on $x + y = 1$. On that line replace $Cx^2 + Dy^2$ by $(Cx^2 + Dy^2)(x + y)$ and gx by $gx(x + y)^2$. The coefficients of x^3, x^2y, xy^2, y^3 are respectively $1, 3, 3, 1$, by (6). Hence $\|P_g\|^2 = (x + y)^3 = 1$ there. As $P_g(0) = 0$, lemma 2.1 proves properness. Joint continuity follows directly from equations (2) and (3).

At $g = 0$ the tuple (A, B, C, D) is $(1/4, 9/4, 3/4, 3/4)$, so

$$P_0(u, v) = \left(0, \frac{\sqrt{3}(v^2 - u^2)}{2}, \frac{u^3 + 3uv^2}{2}, \frac{3u^2v + v^3}{2}\right). \tag{7}$$

Define the unitary maps

$$\begin{aligned} S(u, v) &= \big((u + v)/\sqrt{2}, (u - v)/\sqrt{2}\big), \\ M(a, b, c, d) &= \big(d, -b, (a + c)/\sqrt{2}, (a - c)/\sqrt{2}\big). \end{aligned} \tag{8}$$

The binomial expansion gives $P_0 = M \circ \Phi \circ S$. At $g = 1$, the tuple is $(0, 1, 0, 1)$ and $P_1 = E$. □

Remark 3.2. The four coefficient rows of P_g, in the monomial basis of degree at most three, have disjoint supports

$$\{u\}, \quad \{u^2, v^2\}, \quad \{u^3, uv^2\}, \quad \{u^2v, v^3\}.$$

Their squared lengths are $g, C + D, A + B, A + B$. As $D, B > 0$ on $[0, 1]$, the coefficient Gram rank is three at $g = 0$ and exactly four for $0 < g \leq 1$. No hidden fifth component or compression of a higher-rank squared-norm segment is involved. Convex families of squared norms form a useful general framework [2]; this particular rank-four curve is not the affine chord between its endpoints. Indeed $A(1/2) = 1/6$, whereas the endpoint chord gives $1/8$.

4 Rational untensoring and the endpoint estimate

For $0 \leq a < 1$, put

$$T_a(u, v) = \left(\frac{v - a}{1 - av}, \frac{\sqrt{1 - a^2}\, u}{1 - av}\right), \qquad R_a(u, v) = (u, v^2, uvT_a^1, uvT_a^2). \tag{9}$$

Define the endpoint separately by

$$R_1(u, v) = (u, v^2, -uv, 0). \tag{10}$$

The join is exact: $R_0 = E$.

Proposition 4.1. *The family R_a, $0 \leq a \leq 1$, consists of proper rational maps $\mathbb{B}^2 \to \mathbb{B}^4$ and is jointly continuous on the closed ball. Each slice extends holomorphically past the closed ball. Moreover,*

$$\sup_{\overline{\mathbb{B}^2}} \|R_a - R_1\| \leq 4\sqrt{1 - a}. \tag{11}$$

Proof. For $a < 1$, the denominator in (9) is nonzero on the closed ball because $|av| \le a < 1$. Direct expansion gives

$$1 - \|T_a(u,v)\|^2 = \frac{(1-a^2)(1-|u|^2-|v|^2)}{|1-av|^2}. \tag{12}$$

On $x + y = 1$, where $x = |u|^2$ and $y = |v|^2$, it follows that

$$\|R_a\|^2 = x + y^2 + xy\,\|T_a\|^2 = x + y^2 + xy = 1.$$

The same boundary identity holds for (10). Every R_a vanishes at zero, so lemma 2.1 applies to every slice.

It remains to prove continuity at $a = 1$, including the boundary corner $(u,v) = (0,1)$. Let $\delta = 1 - a > 0$ and $\rho = |v|$. On the closed ball,

$$|u| \le \sqrt{1-\rho^2}, \qquad |1-av| \ge 1 - a\rho \ge \max\{\delta, 1-\rho\}.$$

Only the last two coordinates vary. For the third,

$$\begin{aligned} |uv(T_a^1 + 1)| &= \frac{\delta |uv(1+v)|}{|1-av|} \\ &\le \frac{2\sqrt{2}\,\delta\sqrt{1-\rho}}{\max\{\delta, 1-\rho\}} \le 2\sqrt{2\delta}. \end{aligned} \tag{13}$$

The last inequality follows by splitting into $1-\rho \le \delta$ and $1-\rho \ge \delta$. When $\rho = 1$, $u = 0$ and the estimate holds directly. For the fourth coordinate,

$$|uvT_a^2| = \frac{\sqrt{1-a^2}|u|^2|v|}{|1-av|} \le 2\sqrt{1-a^2}, \tag{14}$$

since $(1-\rho^2)/(1-a\rho) \le 2$ for $\rho < 1$; again $u = 0$ handles $\rho = 1$. Squaring and adding gives $\|R_a - R_1\|^2 \le 8\delta + 4(1-a^2) \le 16\delta$. This proves (11) and joint closed-ball continuity. At $a = 1$ we use the cancelled polynomial (10), never a boundary value of an undefined quotient. □

5 Completion of the homotopy

Apply the unitary target map $J(a,b,c,d) = (a,d,-c,b)$ to R_1. Its result is $(u,0,uv,v^2)$. For $0 \le b \le 1$, set

$$W_b(u,v) = \big(u, \sqrt{1-b^2}\,v, buv, bv^2\big). \tag{15}$$

On $x + y = 1$,

$$\|W_b\|^2 = x + (1-b^2)y + b^2xy + b^2y^2 = x + y = 1.$$

Thus each W_b is proper by lemma 2.1, and $W_0 = L$, $W_1 = JR_1$. This is the usual linear-to-Whitney path.

Proof of theorem 1.1. Use lemma 2.2 to join Φ to $M\Phi S = P_0$. Follow P_g from $g = 0$ to $g = 1$, then R_a from $a = 0$ to $a = 1$. Traverse a unitary path from R_1 to $JR_1 = W_1$, and finally follow W_b from $b = 1$ to $b = 0$. Reparametrize these five pieces onto successive closed subintervals of $[0,1]$. The endpoints agree exactly, so the pasting lemma gives joint continuity on the closed ball. All pieces are origin-preserving proper rational maps with four target coordinates. The polynomial pieces have degree at most three. In the rational piece the common denominator is $1 - av$ and its four numerators have degree at most three. Unitary changes preserve this degree bound. Each polynomial slice is entire; each rational slice with $a < 1$ is holomorphic past the closed ball, and R_1 is polynomial. □

Corollary 5.1. *The four Faran representatives in [3, Example 3.1], after adjoining a zero coordinate, belong to the same homotopy class through proper rational maps into $\mathbb{B}^4$.*

Proof. The target-four paths in that example join the first three representatives. Theorem 1.1 joins the fourth to the linear representative. $\square$

6 Regularity and reproducibility

The rational segment is not continuous in the C^1 boundary topology. At the boundary point $(0,1)$,

$$\partial_u R_a^3(0,1) = 1 \quad (a < 1), \qquad \partial_u R_1^3(0,1) = -1.$$

This is also a real tangential derivative, as seen on the sphere curve $(u,v) = (h, \sqrt{1-h^2})$ at $h = 0$. The jump is compatible with (11). A common holomorphic extension neighborhood is also unavailable for this path: for $0 < a < 1$, the third component has a genuine pole at $v = 1/a$ whenever $u \neq 0$, and these poles approach the source sphere as a tends to one.

There is nevertheless a continuous finite-dimensional coefficient lift for the entire concatenation. Use denominator 1 on the initial unitary and polynomial pieces. On the rational segment use

$$p_a = \big(u(1-av), v^2(1-av), uv(v-a), \sqrt{1-a^2}u^2 v\big), \qquad q_a = 1 - av.$$

At $a = 1$, retain the unreduced pair $((1-v)R_1, 1-v)$. Keep the denominator $1-v$ through the remaining target-unitary and Whitney pieces, multiplying their polynomial maps by $1-v$. All joins then agree in coefficients, $q(0) = 1$ throughout, and the numerator and denominator degrees are at most three and one. Reduced numerator–denominator pairs are not claimed to vary continuously.

The accompanying reproducibility files check (6), the full Hermitian sphere identities, exact coordinate changes, the rational endpoint cancellation, and the interior-defect identities. The main identities are evaluated in exact rational polynomial arithmetic, not inferred from parameter samples. Separate numerical tests are labelled as regressions. These computations support the displayed algebra; the properness, uniform convergence and concatenation arguments are given in the text and do not depend on numerical sampling.

Generative AI assisted with developing the construction, checking algebraic identities, and preparing the exposition. The author takes responsibility for the statements and proofs. The work has undergone the originating researcher's detailed self-audit, but not independent mathematical review or proof-assistant formalization. This unrefereed preprint states its scope without asserting exhaustive novelty or priority clearance.

References

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