

Central Obstructions and Block-Complement Categories for Schur Covers

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Abstract

We study explicit ways to replace the symmetric-group automorphisms of the category of finite sets and injections by their Schur double covers. If the central involution acts freely on a finite lift fibre of an injection of codimension d , then the fibre cardinality is divisible by $2^{\lfloor d/4 \rfloor + 1}$. Thus free two-sheeted morphism lifts, and uniformly polynomial-sized lift fibres, are impossible. We construct an ordinary category using unordered families of internally ordered four-element complement blocks, and a complex superlinear category using two-element blocks. Their composition laws follow from explicit coherent even lifts; the phase needed for pairs is not a group-theoretic section. Finally, a coherent Clifford quotient of the fully framed model has kernel requiring new relations in every rank $n \geq 3$, with quotient dimension $(n-2)2^{n-2}$. Hence that model is not locally noetherian. Classical Schur, Clifford and spinoriality facts are credited. These results concern specified models; they do not establish a general spin representation-stability theory or settle the broad AIM construction program. This is an unrefereed preprint, with no absolute priority claim.

1 Introduction and scope

The AIM Representation stability problem list, item 3.4, attributed to V. Reiner, asks to modify FI using double covers of symmetric groups [1]. Here FI is the category of finite sets and injections; we use its skeleton on $[n] = \{1, \dots, n\}$, including $[0] = \emptyset$. This is a construction program, not a fixed list of axioms. In particular it does not prescribe two-element morphism fibres, local noetherianity, or a character-stability theorem.

Moeller discusses extending the double covers from the category of finite sets and bijections to FI, in the context of extensions of representation-stable categories [3, Section 6, p. 1725]. Our first result addresses an explicit additional hypothesis: the central involution is free on the finite fibre of every injection under consideration. We do not identify that hypothesis with every fibration allowed by Moeller's discussion.

There are three constructions to distinguish. A *fully framed* morphism remembers an ordering of every complement point. A *block-complement* morphism forgets the ordering of blocks but retains the order inside each block. At the level of sets, coherent block lifts exist uniformly for block size divisible by four. Over $\mathbb{C}$, multiplication of lifts by a scalar phase also permits even block sizes, including two. The resulting categories have separated rank components; neither the four-block category nor the pair-linear category lifts every FI injection.

We give a quantitative fibre obstruction, prove the composition laws for both block constructions, and exhibit a non-finitely-generated submodule of a cyclic representable in the fully framed model. The Schur covers, spin algebras, Clifford realizations, and general spinoriality criterion are classical [2, 4]. The criterion for block lifting is a special case of Ganguly–Spallone; we

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include a direct calculation to fix the compatible lifts used in composition. The elementary two-sheet obstruction and fully framed model also appeared in the machine-generated research aid accompanying the frozen `ulamai/UnsolvedMath` record `AIM-REPRESENTATION_THEORY-0033`; they are rederived here, not presented as new discoveries. The additional work is the fibre divisibility bound, coherent block-complement categories, and framed-kernel growth calculation. A bounded primary-source comparison did not locate these exact categorical statements, but does not certify priority.

2 Schur covers and conventions

Fix $\eta \in \{0, 1\}$. In the real Clifford algebra $\text{Cl}_\eta(\mathbb{R}^n)$, the coordinate generators satisfy

$$e_a^2 = (-1)^\eta, \quad e_a e_b = -e_b e_a \quad (a \neq b).$$

Let $z = -1$ and $u_{ab} = (e_a - e_b)/\sqrt{2}$. The subgroup generated by z and these unit difference vectors is G_n^η , the inverse image of the permutation matrices in the corresponding Pin group. Its projection $\pi_n : G_n^\eta \rightarrow S_n$ has kernel $\{1, z\}$, and $|G_n^\eta| = 2n!$. For $n = 0, 1$ take $G_n^\eta = \{1, z\}$. No universal-cover assertion is intended at these small ranks. Coordinate inclusions give strict compatible homomorphisms $j_{mn} : G_m^\eta \hookrightarrow G_n^\eta$.

Put $t_i = u_{i,i+1}$. The usual Schur relations, in these conventions, are

$$t_i^2 = z^\eta, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad t_i t_j = z t_j t_i \quad (|i - j| > 1). \quad (1)$$

For clarity, $t_i t_{i+1} + t_{i+1} t_i = -(-1)^\eta$; this proves the braid relation by multiplication, and $(t_i t_{i+1})^3 = z^\eta$. Disjoint supports give the last relation. Coxeter word reduction leaves at most two signed lifts of each permutation, whereas the Pin realization has exactly two. Thus (1), with the central involution retained, is a presentation. It also fixes signs independently of notation for the two Schur covers in the literature.

Let

$$T_n^\eta = \mathbb{C}[G_n^\eta]/(z + 1).$$

This is the genuine spin group algebra, of dimension $n!$, with one basis element for each underlying permutation. Its parity is permutation parity. The maps j_{mn} induce injective algebra maps, denoted by the same letter. Choosing cosets of S_m in S_n shows that T_n^η is free as a right T_m^η -module of rank $n!/m!$. All dimensions below are ordinary complex vector-space dimensions, not superdimensions. We suppress η where no confusion is possible. Homogeneous Clifford elements on disjoint supports supercommute; in particular, an even complement element commutes with every source element.

3 A finite-fibre obstruction

Theorem 3.1. *Let $\mathcal{C}$ have the same objects as FI, with $\text{Aut}_{\mathcal{C}}(n) = G_n^\eta$, and let $P : \mathcal{C} \rightarrow \text{FI}$ be the identity on objects and restrict to π_n on automorphisms. Fix an injection $f : [m] \rightarrow [n]$. Suppose its fibre $F = P^{-1}(f)$ is finite and nonempty and z_n acts freely on F by postcomposition. For $d = n - m$,*

$$2^{\lfloor d/4 \rfloor + 1} \mid |F|. \quad (2)$$

Proof. Choose $q = \lfloor d/2 \rfloor$ disjoint transpositions supported outside $f([m])$. Their lifts $a_1, \dots, a_q$ preserve F , satisfy $a_i^2 = z^\eta$, and obey $a_i a_j = z a_j a_i$ for distinct i, j . On the (-1) -eigenspace W of z in $\mathbb{C}[F]$, the resulting operators satisfy

$$A_i^2 = (-1)^\eta I, \quad A_i A_j = -A_j A_i.$$

Freeness of the involution gives $\dim W = |F|/2 > 0$.

Set $r = \lfloor q/2 \rfloor = \lfloor d/4 \rfloor$, and let $H_j = iA_{2j-1}A_{2j}$ for $1 \leq j \leq r$. These are commuting involutions. Decompose W into their simultaneous sign eigenspaces W_ε , where $\varepsilon \in \{\pm 1\}^r$. The invertible map A_{2j-1} reverses the j -th sign and preserves all the others. All 2^r eigenspaces therefore have the same dimension. Hence $2^r \mid \dim W$, proving (2). $\square$

Corollary 3.2. *A central two-sheeted morphism cover of FI, with free central action on every fibre, cannot have the groups G_n^η as its automorphism groups. Removing the empty object does not remove the obstruction. For any fixed source rank, nonempty free-central fibres cannot have a uniform polynomial upper bound in the target rank.*

Proof. Use $n \geq m + 4$ for the two-sheet conclusion, and let $n \rightarrow \infty$ in (2) for the polynomial-growth conclusion. $\square$

At codimension four, two disjoint transposition lifts already suffice: their commutator is z , but an action on a two-point set has abelian image. Theorem 3.1 strengthens that argument quantitatively. The divisor is not claimed to be optimal. Nonfree central actions, infinite fibres, or enlarged object sets lie outside its hypotheses.

4 Coherent lifts of block permutations

Fix $k \geq 1$. Let S_r act on kr letters by permuting consecutive ordered k -blocks, preserving position within each block. Define

$$R_j = \prod_{c=1}^k u_{(j-1)k+c, jk+c} \quad (1 \leq j < r), \quad \sigma_k = (-1)^{\binom{k}{2} + \eta k},$$

where factors are ordered by increasing c . The projection of R_j is the adjacent block transposition.

Lemma 4.1. *The block lifts satisfy*

$$R_j^2 = \sigma_k, \quad (R_j R_{j+1})^3 = \sigma_k, \quad R_i R_j = (-1)^k R_j R_i \quad (|i - j| > 1). \quad (3)$$

Proof. The k factors in one R_j are orthogonal odd vectors, giving the first identity. For the second, write $a_c = u_{(j-1)k+c, jk+c}$ and $b_c = u_{jk+c, (j+1)k+c}$. Then

$$R_j R_{j+1} = (-1)^{\binom{k}{2}} \prod_{c=1}^k (a_c b_c).$$

The even factors $a_c b_c$ commute for different c , and $(a_c b_c)^3 = (-1)^\eta$. Cubing gives the second identity. For disjoint block transpositions, moving k odd factors past k others gives $(-1)^{k^2} = (-1)^k$. $\square$

Proposition 4.2 (Classical lifting criterion, with coherent choices). *The block embedding $S_r \hookrightarrow S_{kr}$ lifts to G_{kr}^η for every r if and only if $4 \mid k$. For $4 \mid k$, the lifts $s_j \mapsto R_j$ define sections coherent under adding blocks and under consecutive block sums. Their images are even and do not contain z . Necessity already follows from $r = 4$.*

Proof. For odd k , lifts of two disjoint block transpositions anticommute. Changing their signs does not change their commutator, so the commuting generators s_1, s_3 of S_4 cannot lift. For $k \equiv 2 \pmod{4}$, every lift of a block transposition squares to z , and hence cannot be an involution.

If $4 \mid k$, (3) gives the Coxeter relations for S_r . The resulting homomorphism projects to its faithful block action, so is injective and excludes z . Each generator is even. Its coordinate formula is unchanged by a coordinate offset or by adding consecutive blocks. This is the required coherence, including the embeddings of each factor under a consecutive block sum. $\square$

The criterion itself is not new. For k copies of the permutation representation of S_r , the character differences between the identity and a transposition or two disjoint transpositions are $2k$ and $4k$. Ganguly–Spallone’s criterion [2, Theorem 1.1], in their negative-square Pin convention, gives exactly $4 \mid k$. The direct proof above treats both signs and supplies the chosen compatible lifts needed below. It does not classify exceptional lifts at individual small r .

5 An ordinary block-complement category

Fix a positive multiple k of four. If $n - m = kr \geq 0$, let $H_{mn} \leq G_n$ be the section of S_r from Proposition 4.2, acting on the last kr coordinates and fixing the first m . Put

$$\mathcal{C}_k(m, n) = \begin{cases} G_n/H_{mn}, & n - m \in k\mathbb{Z}_{\geq 0}, \\ \emptyset, & \text{otherwise.} \end{cases}$$

These are right cosets. For $x \in G_n$, $y \in G_p$, define

$$[y] \circ [x] = [y j_{np}(x)]. \quad (4)$$

Theorem 5.1. *Formula (4) defines a category with $\text{Aut}(n) = G_n$, a free central involution on every nonempty Hom-set, and a functor $P : \mathcal{C}_k \rightarrow \text{FI}$. Modulo the central involution, a morphism is an injection together with a partition of its complement into an unordered family of internally ordered k -tuples. For $r = (n - m)/k \geq 0$,*

$$|\mathcal{C}_k(m, n)| = \frac{2n!}{r!}, \quad |P^{-1}(f)| = \frac{2(n - m)!}{r!}.$$

The genuine complex linear Hom-space has dimension $n!/r!$.

Proof. There are two representative changes to check. Replacing x by xh , with $h \in H_{mn}$, introduces the right factor $j_{np}(h)$, which lies in H_{mp} by consecutive-block coherence. Replacing y by yb , with $b \in H_{np}$, gives $ybj_{np}(x)$. Since b is even and supported beyond the first n coordinates, it commutes with $j_{np}(x)$. It also belongs to H_{mp} . Thus (4) is well-defined. Strict inclusions and associative multiplication prove associativity; $[1]$ is the identity.

The subgroup H_{nn} is trivial, so the endomorphisms are precisely G_n , all invertible. If $z[g] = [g]$, centrality would imply $z \in H_{mn}$, contrary to Proposition 4.2. This proves the free central action.

Define $P([g]) = \pi_n(g)|_{[m]}$. It is independent of the representative, since H_{mn} fixes $[m]$, and it respects composition. The images of the remaining coordinates encode an ordered list of internally ordered blocks; quotienting by H_{mn} forgets only the order of these blocks. Quotienting by z forgets the lift sign. This proves the claimed description.

The subgroup has order $r!$, yielding the total cardinality. There are $n!/(n - m)!$ underlying injections, each with the same fibre size. Finally, in the linearization with $z = -1$, each free pair of basis elements contributes one dimension. $\square$

In particular, $\mathcal{C}_4$ has four components indexed by $n \bmod 4$. Minimal block size four refers only to this uniform set-level block model. We claim neither local noetherianity nor representation stability for $\mathcal{C}_4$.

6 Complex linear pairs

Now let k be even, so $\sigma_k = (-1)^{k/2}$ is independent of η . Set

$$\alpha = \begin{cases} 1, & k \equiv 0 \pmod{4}, \\ i, & k \equiv 2 \pmod{4}, \end{cases} \quad B_j = \alpha R_j \in T_{kr}.$$

Here $\alpha^2 = \sigma_k$. Equation (3) gives

$$B_j^2 = 1, \quad (B_j B_{j+1})^3 = 1, \quad B_i B_j = B_j B_i \quad (|i - j| > 1).$$

Thus the B_j define an even copy $B_r \cong \mathbb{C}[S_r]$ inside T_{kr} . Indeed every permutation maps to a nonzero scalar multiple of the spin-algebra basis vector of its block permutation; distinct permutations have distinct support. These copies are coherent under consecutive block sums, with the same phase after every coordinate offset.

For $n - m = kr \geq 0$, write B_{mn} for the corresponding complement algebra and I_{mn} for the left ideal of T_n spanned by $a(b - 1)$, with $a \in T_n$ and b in the specified group basis of B_{mn} . Define

$$\mathcal{L}_k(m, n) = \begin{cases} T_n/I_{mn}, & n - m \in k\mathbb{Z}_{\geq 0}, \\ 0, & \text{otherwise,} \end{cases} \quad [y] \circ [x] = [y j_{np}(x)].$$

Theorem 6.1. *For every positive even k , these formulas define a complex superlinear category with $\text{End}(n) = T_n$ and $\dim \mathcal{L}_k(m, n) = n!/r!$, where $r = (n - m)/k \geq 0$. In particular $k = 2$ is allowed. Rescaling the standard block lifts by nonzero scalars cannot yield this uniform construction for odd k .*

Proof. Every $b \in B_{np}$ is even and supported beyond n , so commutes with $j_{np}(T_n)$. Changing y by a generator $a(b - 1)$ of I_{np} changes the composite by $a j_{np}(x)(b - 1)$, in I_{mp} . Changing x by $a(b - 1)$, with $b \in B_{mn}$, also gives an element of I_{mp} , because $j_{np}(B_{mn}) \subset B_{mp}$. Bilinearity now handles both changes. The ideals are homogeneous, as the elements b are even. Associativity, parity compatibility and identities descend from multiplication and strict inclusions. At $m = n$, the ideal is zero.

Choose representatives for the right cosets of the block-permutation subgroup S_r in S_n . Their spin lifts, multiplied by the basis elements of B_{mn} , give a basis of T_n : every product has nonzero coefficient and a distinct underlying permutation. Consequently T_n is free as a right B_{mn} -module of rank $n!/r!$, and

$$T_n/I_{mn} \cong T_n \otimes_{B_{mn}} \mathbb{C},$$

where $\mathbb{C}$ is the trivial B_{mn} -module. This proves the dimension.

For odd k , arbitrary nonzero scalar multiples of R_1 and R_3 still anticommute. They cannot represent the commuting generators of S_4 . $\square$

For $k = 2$, the scalar i is essential and is not a sign in the real Schur cover. Thus this is not a group section or the linearization of a free two-sheeted set cover. There are still two rank components, indexed by parity; no stability assertion is implicit in the construction. The quotients for $m < n$ are Hom-spaces, not asserted to be quotient algebras.

7 Failure of local noetherianity for full framing

Let $\mathcal{F}$ be the complex superlinear category

$$\mathcal{F}(m, n) = T_n \quad (m \leq n), \quad y \circ x = y j_{np}(x),$$

and let $\mathcal{F}(m, n) = 0$ for $m > n$. A module means a covariant linear functor, with the analogous graded definition when parity is retained. The representable $P_0 = \mathcal{F}(0, -)$ is cyclic, generated by its unit at rank zero; $P_0(n) = T_n$.

First use $\eta = 0$. For $n \geq 1$, let $U_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : \sum x_i = 0\}$, with its nondegenerate standard symmetric form. The assignment

$$q_n : T_n \twoheadrightarrow \text{Cl}(U_n), \quad t_i \mapsto (e_i - e_{i+1})/\sqrt{2}$$

is a surjective algebra homomorphism. The relations hold by Section 2; the difference vectors span U_n and so generate its Clifford algebra. Furthermore,

$$\dim \text{Cl}(U_n) = 2^{n-1}. \quad (5)$$

For example, ordered squarefree monomials span by the Clifford relations; the representation on $\bigwedge U_n$ by exterior multiplication plus contraction, evaluated at 1, proves independence by highest exterior degree. Coordinate inclusions $U_m \hookrightarrow U_n$ induce injective Clifford maps and make the q_n coherent. Take $q_0 : \mathbb{C} \rightarrow \mathbb{C}$ to be the identity.

Theorem 7.1. *The spaces $J_n = \ker q_n$ form a graded subfunctor J of P_0 , which is not finitely generated. For every $n \geq 2$,*

$$\dim(J_n/T_n J_{n-1}) = (n-2)2^{n-2}. \quad (6)$$

In particular $\mathcal{F}$ is not locally noetherian, for either Schur sign over $\mathbb{C}$.

Proof. Coherence identifies J_m with a subspace of J_n for $m \leq n$. Since J_n is a left ideal, arbitrary left morphism actions preserve these kernels. Thus J is a graded subfunctor.

The right T_{n-1} -module T_n is free of rank n . Tensoring the injection $J_{n-1} \hookrightarrow T_{n-1}$ is therefore exact; multiplication identifies the resulting subspace with $T_n J_{n-1}$. Using (5) gives

$$\dim T_n J_{n-1} = n((n-1)! - 2^{n-2}), \quad \dim J_n = n! - 2^{n-1}.$$

Their difference is (6).

This difference is positive for every $n \geq 3$. Images of all lower-rank kernels are contained in $T_n J_{n-1}$, because the kernels are nested and $T_n J_{n-1}$ is a left ideal. Hence new relations are needed in unbounded ranks. A finite generating set has bounded ranks, so cannot generate J . We have found a non-finitely-generated submodule of the cyclic module P_0 , proving the failure of local noetherianity. For the other Schur sign, rescaling every adjacent generator by i gives a coherent isomorphism of the complex spin-algebra towers and transfers the conclusion. $\square$

The quantity in (6) is a quotient vector-space dimension, not the minimum number of T_n -module generators. The theorem applies to full framing only. Neither $\mathcal{C}_4$ nor $\mathcal{L}_2$ inherits this conclusion without another argument.

8 Verification and remaining questions

The accompanying standard-library Python checker uses exact rational arithmetic in a sparse Clifford basis over $\mathbb{Q}(\sqrt{2})$. It checks block square, adjacent-cube and disjoint-commutator identities for $k = 1, \dots, 4$, in both Pin conventions; finite coherent sections through four blocks; and their interactions with disjoint source lifts. It independently computes signed permutation lifts through rank six, the Clifford image rank 2^{n-1} , and induced kernel-ideal ranks for $n = 3, 4, 5$. It also checks the phase repair for pairs and the numerical dimension identities. There are 14,849 exact conditions in each run; normal and optimized interpreter executions have identical output. These finite tests supplement the all-rank proofs. They are neither proof-assistant verification nor independent peer review.

The construction program in the AIM source remains open beyond the precise models treated here. Natural next questions include sharp finite-fibre bounds, local noetherianity of the block-complement categories, and useful character or representation-stability statements. Existence of the present categories does not prove those properties.

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