A Degree-24 Obstruction to Hopf Structures on Symmetric-Group Supercharacter Spaces

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Abstract

Consider the supercharacter theory of S_n whose superclasses are the S_n-conjugacy classes contained in A_n, together with the single block $S_n \setminus A_n$. We prove that its superclass-function spaces, summed over n with one-dimensional components in degrees zero and one, admit no connected graded Hopf algebra structure over a field of characteristic zero. The argument uses a known nonnegative Euler-product condition for Hopf Hilbert series. The first negative Euler exponent occurs in degree 24 and equals -1: lower degrees force 795 basis monomials, but the required dimension is 794. The two coarser natural supercharacter families fail the same test in degree 4. These obstructions exclude arbitrary graded Hopf operations, not only quotients of symmetric functions. We state the families explicitly and do not claim a classification of all symmetric-group supercharacter theories.

1 The families and the result

Work over a field k of characteristic zero and in the ordinary, unsigned category of Hopf algebras. A connected graded structure means an $\mathbb{N}$-grading compatible with all the Hopf operations and with degree-zero space k. For a superclass partition $\mathcal{K}_n$ of S_n, let $V_n = k^{\mathcal{K}_n}$ be its space of superclass functions. Put $V_0 = V_1 = k$. For $n \geq 2$ we consider these partitions:

- the *fine* partition: the individual S_n-conjugacy classes in A_n, and the block $S_n \setminus A_n$;
- the *coarse* partition: $\{1\}$, $A_n \setminus \{1\}$, and $S_n \setminus A_n$, with empty blocks omitted;
- the *maximal* partition: $\{1\}$ and $S_n \setminus \{1\}$.

The first two are Hendrickson star-products over $A_n \triangleleft S_n$: use, respectively, the finest S_n-invariant theory or the maximal theory on A_n, and the unique theory of $S_n/A_n \cong C_2$ [2, Theorem 5.3 and pp. 8–10 of the preprint].

Theorem 1. *For each of the fine, coarse and maximal families, $\bigoplus_{n\geq 0} V_n$ admits no connected graded Hopf algebra structure preserving degree n. The first negative Euler exponents occur in degrees* 24, 4 *and* 4*, respectively. For the ordinary conjugacy-class family, the corresponding graded space is the classical positive case* Sym*.*

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2 The Hilbert-series criterion and the dimensions

We use the following known consequence of the structure theory of connected graded Hopf algebras.

Lemma 2 ([3, Proposition 3.5]). *If H is a connected, locally finite, graded Hopf algebra over k, then*

$$\mathrm{Hilb}_H(t) = \prod_{r\geq 1}(1-t^r)^{-a_r} \quad \textit{for integers } a_r \geq 0.$$

The exponents are uniquely determined by the Hilbert series.

Explanation of the criterion. The homogeneous PBW basis in [3, Theorem A] consists of ordered monomials in positive-degree elements. If a_r counts the elements of degree r, local finiteness makes a_r finite, and counting monomials gives the product. Uniqueness follows successively: after the factors below degree n are fixed, insertion of $(1-t^n)^{-a_n}$ increases the coefficient of t^n by exactly a_n. Higher-degree factors have no effect on that coefficient. □

No commutativity, cocommutativity, finite Gelfand–Kirillov dimension, or algebraic closedness assumption is needed. In particular, the criterion does not assert that the original Hopf algebra is polynomial. The cited paper credits the Hilbert-series result to Brown, Gilmartin and Zhang.

Let $p(n)$ count partitions of n, and let $s(n)$ count partitions into distinct odd parts. A permutation of cycle type $\lambda \vdash n$ has sign $(-1)^{n-\ell(\lambda)}$. If $E(n)$ counts the even cycle types, then

$$\begin{aligned}\sum_{n\geq 0}(2E(n)-p(n))t^n &= \prod_{j \text{ odd}} \frac{1}{1-t^j} \prod_{j \text{ even}} \frac{1}{1+t^j} \\ &= \prod_{j \text{ odd}} (1+t^j).\end{aligned}$$

For the second equality, use $(1+t^{2j})^{-1} = (1-t^{2j})/(1-t^{4j})$ and cancel the factors indexed by multiples of four. All products are formal. Thus $E(n) = (p(n)+s(n))/2$. For $n \geq 2$ the fine partition has $E(n)$ even-class blocks and one odd block, so its dimensions are

$$h_0 = h_1 = 1, \qquad h_n = \frac{p(n)+s(n)}{2} + 1 \quad (n \geq 2). \tag{1}$$

Its full Hilbert series is therefore

$$H_f(t) = \frac{1}{2}\left(\prod_{j\geq 1}\frac{1}{1-t^j} + \prod_{j \text{ odd}}(1+t^j)\right) + \frac{t^2}{1-t}. \tag{2}$$

The coarse and maximal dimension sequences are, respectively,

$$(1,1,2,3,3,\ldots), \qquad (1,1,2,2,2,\ldots).$$

In particular the exceptional coarse value at $n = 2$ is 2, since A_2 is trivial. These low-degree conventions are essential to the calculation.

3 The finite obstruction

For any $H(t) = \sum_{n\geq 0} h_n t^n$ with $h_0 = 1$, define $tH'(t)/H(t) = \sum_{n\geq 1} b_n t^n$. Formal differentiation and coefficient comparison give the exact recurrences

$$b_n = nh_n - \sum_{j=1}^{n-1} b_j h_{n-j}, \qquad a_n = \frac{1}{n}\left(b_n - \sum_{\substack{d|n \\ d<n}} d a_d\right). \tag{3}$$

Using (1) gives Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12
h_n	1	2	3	4	5	7	9	13	17	23	30	41
a_n	1	1	1	0	0	0	1	2	1	1	1	3
n	13	14	15	16	17	18	19	20	21	22	23	24
h_n	53	70	91	119	152	196	249	318	401	506	633	794
a_n	3	2	2	2	3	3	3	2	1	1	1	−1

Table 1: Fine-family dimensions and their unique Euler exponents.

For a direct multiplication check, form

$$C(t) = \prod_{r=1}^{23} (1 - t^r)^{-a_r}$$

with the nonnegative exponents in the table. Truncation gives

$$H_f(t) - C(t) \equiv -t^{24} \pmod{t^{25}}. \tag{4}$$

In particular $[t^{24}]C(t) = 795$, whereas

$$h_{24} = \frac{1575 + 11}{2} + 1 = 794.$$

The coefficient lists in the table, together with either (3) or (4), are a finite certificate of this one-dimensional deficit. A degree-24 generator can only increase the forced value 795. Lemma 2 therefore rules out every putative graded Hopf structure for the fine family. Notice also that $C(t)$ is realized by a polynomial Hopf algebra on primitive generators of the listed degrees: the whole prefix through degree 23 is actually compatible with a Hopf Hilbert series, not merely with a formal necessary inequality.

For the coarse family the first four exponents are $(1, 1, 1, -1)$, and for the maximal family they are $(1, 1, 0, -1)$. Their negative fourth exponents give the other two obstructions in Theorem 1. The ordinary family has $h_n = p(n)$ and every exponent equal to one, as for Sym.

The source package contains an exact Python verifier. Generating-function counts are compared with recursive enumeration of cycle types, and logarithmic-derivative inversion is compared with successive Euler-factor multiplication. All 333 checks pass on two Python versions, including a run with assertions disabled. These checks verify the finite arithmetic, not the cited structure theorem or novelty.

4 Relation to the source question

Problem 1.18 in the 2010 AIM list [1] asks about Hopf structures and quotients of Sym for four unnamed supercharacter families. The adjacent Problem 1.17 asks a classification question. Ladisch's later Conjecture 6.2 [4] records the four-theory conjecture for $n \geq 7$ and notes the two extra constructions from A_n. We have defined the families explicitly; we do not prove that there are no other families or exceptional theories.

For a graded Hopf quotient of Sym in characteristic zero, every Euler exponent is zero or one. Hence the fine family's $a_8 = 2$ already excludes that narrower construction. The degree-24 calculation strengthens this to an obstruction against arbitrary graded multiplication and comultiplication. The result does not cover a changed grading, positive characteristic, or the Koszul-signed category. In particular, signed exterior Hopf algebras illustrate why the unsigned convention cannot be discarded.

This is an unrefereed preprint. The general obstruction and the superclass constructions are known; the specific finite application is the contribution proposed here. A bounded literature search is not an absolute-priority certificate. AI-assisted tools supported research, exact computation and writing; the author is responsible for the final text.

References

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