

Quantum Symmetric Algebras on Multiple Standard Copies: Structure and Categorical Non-Equivalence

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Abstract

Let E be the standard object in the generic polynomial Hecke category over $\mathbb{C}(q)$, and let U be an ordinary finite-dimensional multiplicity space. We determine every homogeneous component of the Berenstein–Zwacknagl quantum symmetric algebra $A = S_\sigma(E \otimes U)$: in degree $n \geq 2$, only the one-row and one-column representations survive, with multiplicities $\text{Sym}^n U$ and $\bigwedge^n U$. We identify A as a fiber product of two diagonal algebras over its square-zero degree-one truncation. For $\dim U \geq 2$, its full category of internal polynomial right modules is not equivalent to the corresponding classical module category, even as an abstract abelian category. The obstruction is intrinsic: the directed extension graph identifies two simple objects whose projective covers have a zero Hom space quantumly and a nonzero Hom space classically. The calculation uses an explicit two-dimensional Hecke braid defect and does not assume that an equivalence preserves an augmentation, grading, tensor action, or finite-rank evaluation.

1 Introduction and scope

Berenstein and Zwacknagl define symmetric and exterior algebras using the normalized quantum commutator [3]. These constructions need not be flat deformations of classical symmetric and exterior algebras. For the direct sum of two standard two-dimensional modules, nonflatness already occurs in degree three; Tucker-Simmons gives an explicit calculation in [4, Example 3.5.1.5, pp. 74–75]. We do not claim that nonflatness observation as new.

Problem 1.8 of the AIM representation-stability list, proposed by D. Barter, asks about equivalence between quantum and classical tca module categories; the accompanying remark singles out the doubled standard object [1]. Barter proves a positive result for the single-standard quantum symmetric algebra through the horizontal-strip category [2]. A loss of dimension under a particular evaluation does not, by itself, obstruct an arbitrary equivalence of internal module categories. Our purpose is to supply such an intrinsic obstruction and to calculate the full algebra on any finite number of standard copies.

Throughout, $k = \mathbb{C}(q)$, with q indeterminate. All module categories are the full categories of internal polynomial *right* modules in the Ind-completions specified below. We do not pass to a quotient by torsion, restrict to ordinary unequivariant modules after evaluation, or assert a classification for arbitrary quantum tca generators.

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Theorem 1.1. *Let $U = k^m$, $m \geq 1$, and $A = S_\sigma(E \otimes U)$ in $\text{Ind}(\mathcal{C}_q)$. Let $P = \text{Sym}(E \otimes U)$ in the corresponding classical category. Then:*

1. *For every $n \geq 2$,*

$$A_n \cong L_{(n)} \otimes \text{Sym}^n U \oplus L_{(1^n)} \otimes \bigwedge^n U. \quad (1)$$

The components $A_0 = k$ and $A_1 = E \otimes U$ are stated separately.

2. *Put*

$$B^+ = \bigoplus_{n \geq 0} \text{Sym}_q^n(E) \otimes \text{Sym}^n U, \quad B^- = \bigoplus_{n \geq 0} \bigwedge_q^n(E) \otimes \bigwedge^n U.$$

With their diagonal algebra structures and truncation maps to $T = k \oplus (E \otimes U)$, where $(E \otimes U)^2 = 0$, there is a natural isomorphism of graded algebra objects

$$A \cong B^+ \times_T B^-. \quad (2)$$

3. *If $m \geq 2$, the abelian categories $\text{Mod}_{\text{Ind}(\mathcal{C}_q)}(A)$ and $\text{Mod}_{\text{Ind}(\mathcal{C}_{\text{cl}})}(P)$ are not equivalent. No k -linearity or compatibility with additional structures is required of a proposed equivalence.*

The non-equivalence uses only the degree-three part of equation (1); it does not depend on the fiber-product assertion. The parameter $q = 1$ is excluded from the theorem, and no root-of-unity statement is asserted. The single-copy case $m = 1$ does not yield the obstruction in part (3), consistently with [2].

2 The generic Hecke category and the quadratic quotient

For $n \geq 0$, let $H_n(q)$ be the type- A Hecke algebra over k , with adjacent generators g_i satisfying the braid relations and

$$(g_i - q)(g_i + q^{-1}) = 0, \quad g_i^2 = (q - q^{-1})g_i + 1.$$

We take $H_0(q) = H_1(q) = k$. Start with the monoidal category whose objects are nonnegative integers, whose endomorphism algebra at n is $H_n(q)$, and whose morphisms between distinct integers vanish. Tensor product on objects is addition and on endomorphisms is the standard inclusion $H_a(q) \otimes H_b(q) \rightarrow H_{a+b}(q)$. Its additive idempotent completion is $\mathcal{C}_q$. Replace $H_n(q)$ by $k[\mathfrak{S}_n]$ to define $\mathcal{C}_{\text{cl}}$. These are the polynomial categories used in the description of [2, Sections 2–4].

Generic Hecke semisimplicity gives simple objects L_λ , indexed by partitions, with $\text{End}(L_\lambda) = k$ and polynomial degree $|\lambda|$. Tensor products have the usual Littlewood–Richardson multiplicities. In particular,

$$L_\lambda \otimes E \cong \bigoplus_{\mu: \mu/\lambda \text{ is one box}} L_\mu, \quad E = L_{(1)}. \quad (3)$$

We use the same labels in $\mathcal{C}_{\text{cl}}$, without identifying the two categories as monoidal categories. The Ind-completions allow arbitrary direct sums of these simple polynomial objects. Kernels, cokernels and polynomial degree components are computed componentwise.

An expression $X \otimes U$, for an ordinary vector space U , means the external multiplicity object, a direct sum of copies of X . It introduces no second quantum-group action. On $E \otimes E$ the normalized commutator is the involution

$$s = \frac{2g - (q - q^{-1})1}{q + q^{-1}}, \quad (4)$$

acting by $+1$ on $L_{(2)}$ and by -1 on $L_{(1,1)}$. For $W = E \otimes U$, naturality on the multiplicity space gives $\sigma_W = s \otimes \tau$, after regrouping, where τ is the usual flip of $U \otimes U$. Thus

$$A = T(W)/\langle R \rangle, \quad R = \ker(1 + \sigma_W) = \bigwedge_q^2 E \otimes \text{Sym}^2 U \oplus \text{Sym}_q^2 E \otimes \bigwedge^2 U. \quad (5)$$

This is the normalized-commutator construction of [3], not the flat quantum matrix algebra obtained by deforming both tensor factors.

Write $V_\lambda = \text{Hom}_{\mathcal{C}_q}(L_\lambda, E^{\otimes n})$ for the Hecke multiplicity module in degree $n = |\lambda|$. Each adjacent g_i acts on this module; write s_i for its normalization equation (4), and τ_i for the adjacent permutation on $U^{\otimes n}$. The L_λ multiplicity space of A_n is

$$M_\lambda = \frac{V_\lambda \otimes U^{\otimes n}}{\sum_{i=1}^{n-1} \text{im}(1 - s_i \otimes \tau_i)}. \quad (6)$$

Indeed, the degree- n part of the quadratic ideal is $\sum_i W^{\otimes(i-1)} \otimes R \otimes W^{\otimes(n-i-1)}$. Because σ_W is an involution in characteristic zero, its negative eigenspace equals $\text{im}(1 - \sigma_W)$. Taking a simple multiplicity is exact in the semisimple ambient category, which proves equation (6). No invariance of that relation subspace under all of $H_n(q)$ is being assumed.

3 A local braid defect and its propagation

Lemma 3.1. *On the two-dimensional simple $H_3(q)$ -module, the operator $(s_1 s_2)^3 - 1$ is invertible.*

Proof. Set $D = q^2 + 1$, $F = q^4 + q^2 + 1$, and $a = (q^4 + 1)/D^2$. A model is

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -a & 1 \\ 1 - a^2 & a \end{pmatrix}.$$

They are involutions. For $g_i = ((q + q^{-1})s_i + (q - q^{-1})1)/2$, direct multiplication verifies the Hecke quadratic relation and $g_1 g_2 g_1 = g_2 g_1 g_2$. The representation is irreducible: neither of the two eigenspaces of s_1 is stable under s_2 , since

$$1 - a^2 = \frac{4q^2 F}{D^4} \neq 0.$$

It is therefore the unique two-dimensional simple, of label $(2, 1)$. For $C = s_1 s_2$, we have $\det C = 1$ and $\text{tr } C = t = -2a$. Cayley–Hamilton gives $\text{tr}(C^3) = t^3 - 3t$, whence

$$\begin{aligned} \det(C^3 - 1) &= 2 - \text{tr}(C^3) = -(t - 2)(t + 1)^2 \\ &= \frac{4(q^4 + q^2 + 1)(q^2 - 1)^4}{(q^2 + 1)^6} \neq 0. \end{aligned} \quad (7)$$

All equalities are identities over k . Transposing the representation or reversing $s_1 s_2$ preserves this invertibility: the reversed product is conjugate to C . $\square$

Lemma 3.2. *Let V be a simple $H_n(q)$ -module with $n \geq 3$. If $v \in V$ satisfies $g_i v = g_j v$ for all i, j , then either $v = 0$ or V is the one-row or the one-column representation.*

Proof. Write $w = g_i v$, independent of i . The quadratic relation gives $g_i w = (q - q^{-1})w + v$ for every i . Consequently $\text{span}(v, w)$ is a submodule on which all generators act by the same operator. Its minimal polynomial divides $(z - q)(z + q^{-1})$, which has distinct roots in k . It is therefore a sum of one-dimensional modules, with all generators acting by q or all acting by $-q^{-1}$. If $v \neq 0$, simplicity makes V one of these modules. $\square$

Proposition 3.3. *Equation (1) holds for every $n \geq 2$.*

Proof. All multiplicity spaces in equation (6) are finite-dimensional. An element $f \in M_\lambda^*$ satisfies $(u_i \otimes v_i)f = f$, where $u_i = s_i^*$ and $v_i = \tau_i^*$. Since both factors are involutions,

$$(u_i \otimes 1)f = (1 \otimes v_i)f. \quad (8)$$

Moving successive operators between the factors reverses the word: for any adjacent indices $i, i+1$,

$$((u_i u_{i+1})^3 \otimes 1)f = (1 \otimes (v_{i+1} v_i)^3)f = f.$$

The last equality is an ordinary symmetric-group relation.

Restrict V_λ^* to the parabolic $H_3(q)$ on g_i, g_{i+1} . This restriction is semisimple. On every two-dimensional constituent, $(u_i u_{i+1})^3 - 1$ is invertible by lemma 3.1. On the two one-dimensional constituents it is zero, and $u_i - u_{i+1}$ is zero there as well. It follows that $((u_i - u_{i+1}) \otimes 1)f = 0$. Using equation (4), we also have $((g_i^* - g_{i+1}^*) \otimes 1)f = 0$. Expand f in a basis of $(U^{\otimes n})^*$. Each coefficient in V_λ^* has the same image under every g_i^* . The transposed generators satisfy the same defining Hecke relations, so lemma 3.2 applies to this simple dual module. If λ is neither a row nor a column, all coefficients vanish. Thus $M_\lambda = 0$.

On the row representation all $s_i = 1$. Equation (6) is then the usual symmetric quotient $\text{Sym}^n U$. On the column representation all $s_i = -1$, and it is the exterior quotient $\bigwedge^n U$. For $n = 2$ these are already the only two labels, so the same statement follows directly from equation (5). The cases $n = 0, 1$ are the unit and generating object, respectively. $\square$

4 The fiber product and finite-rank dimensions

The standard quantum symmetric and exterior algebras on E have degree- n components $L_{(n)}$ and $L_{(1^n)}$. This also follows by taking the quotient of $E^{\otimes n}$ imposing, respectively, $g_i = q$ and $g_i = -q^{-1}$ for all adjacent generators. For $B^\pm$ in theorem 1.1, multiply in both factors and take the diagonal grading. The multiplicity factors use ordinary multiplication: there is no additional super sign or second quantum deformation. Both algebras are generated by $W = E \otimes U$. Indeed, products of n pairs of degree-one generators span the tensor product of the two degree- n factors.

Proposition 4.1. *The degree-one identity induces the isomorphism equation (2).*

Proof. The maps from the tensor algebra $T(W)$ to B^+ and B^- kill both summands of R in equation (5). For B^+ , the first summand is killed by symmetric multiplication on E , and the second by symmetric multiplication on U . For B^- , exterior multiplication on U kills the first summand, and exterior multiplication on E kills the second. The two maps agree after truncation to $T = k \oplus W$. They therefore define an algebra map $\phi : A \rightarrow B^+ \times_T B^-$.

In degrees zero and one this is the diagonal identity. In each degree $n \geq 2$, the target is $B_n^+ \oplus B_n^-$. Both projections of the image are surjective because $B^\pm$ are generated in degree one. Their simple supports, (n) and (1^n) , are distinct. In a semisimple category, a subobject of two objects with disjoint simple supports that surjects to each is their whole direct sum. Hence ϕ_n is surjective. By proposition 3.3, its source and target have the same finite simple multiplicities; thus it is an isomorphism. $\square$

Corollary 4.2. *If $m = 2$, the kernel of $A \rightarrow B^+$ is $\bigwedge_q^2 E \otimes \bigwedge^2 U$, concentrated in degree two. It is annihilated on both sides by A_+ , and is square-zero.*

Proof. For $n > 2$ the factor $\bigwedge^n U$ vanishes. The claim follows from the coordinatewise multiplication of the fiber product. No splitting of $A \rightarrow B^+$ as algebras is asserted. $\square$

Under standard rank- d quantum Schur–Weyl evaluation, the two row/column objects have the classical dimensions of the symmetric/exterior powers of a d -dimensional vector space. Exactness of this evaluation in the generic semisimple category gives, for $n \geq 2$,

$$\dim A_n^{(d)} = \binom{d+n-1}{n} \binom{m+n-1}{n} + \binom{d}{n} \binom{m}{n}. \quad (9)$$

Here $\binom{r}{n} = 0$ for $n > r$. In particular, for $d = m = 2$ the Hilbert series is

$$1 + 4t + 10t^2 + \sum_{n \geq 3} (n+1)^2 t^n = \frac{1+t}{(1-t)^3} + t^2.$$

This evaluation is a consequence and a check, not a compatibility condition imposed on an equivalence of module categories.

5 An intrinsic categorical obstruction

We give the categorical part in a form that applies to both A and P . Let B denote either algebra and $\mathcal{C}$ its corresponding polynomial category. Both $B_0 = k$ and $B_1 = E \otimes U$, and B is generated in degree one.

Lemma 5.1. *The simple objects of $\text{Mod}_{\text{Ind}(\mathcal{C})}(B)$ are precisely the augmentation modules L_λ . Each has projective cover $Q_\lambda = L_\lambda \otimes B \rightarrow L_\lambda$.*

Proof. For a nonzero simple module M , let d be its least nonzero polynomial degree. The sum $M_{>d}$ is a proper submodule: the algebra has no negative degrees, so multiplication cannot lower degree. Simplicity forces $M_{>d} = 0$. Therefore B_+ acts trivially. Every ambient subobject of M_d is now a submodule, which forces $M = L_\lambda$ with the augmentation action. Conversely these modules are simple.

The free right-module adjunction is

$$\text{Hom}_B(L_\lambda \otimes B, M) = \text{Hom}_{\mathcal{C}}(L_\lambda, M).$$

The forgetful functor is exact, and the right-hand functor is exact in the semisimple ambient category. Thus Q_λ is projective. If a submodule $N \subset Q_\lambda$ surjects onto L_λ under augmentation, its degree- $|\lambda|$ part contains the entire generator L_λ , since augmentation is the identity there. That generator generates all of Q_λ , so $N = Q_\lambda$. The kernel of augmentation is consequently superfluous, proving the projective-cover assertion. $\square$

Projective covers and their uniqueness up to isomorphism are categorical. For completeness, maps between two covers lifting the identity of the simple have composites with full image by superfluity; their epimorphic endomorphisms split by projectivity. A nonzero complementary kernel would contradict superfluity again, so these composites, and then the lifting maps, are isomorphisms.

Lemma 5.2. *For simple augmentation modules,*

$$\mathrm{Ext}_B^1(L_\lambda, L_\mu) \cong \mathrm{Hom}_C(L_\lambda \otimes (E \otimes U), L_\mu).$$

In particular this group is nonzero exactly when μ is obtained from λ by adding one box.

Proof. Apply $\mathrm{Hom}_B(-, L_\mu)$ to

$$0 \longrightarrow L_\lambda \otimes B_+ \longrightarrow Q_\lambda \longrightarrow L_\lambda \longrightarrow 0.$$

Projectivity of Q_λ gives $\mathrm{Ext}_B^1(L_\lambda, L_\mu) \cong \mathrm{Hom}_B(L_\lambda \otimes B_+, L_\mu)$: the preceding restriction map is zero since every map to an augmentation module kills $Q_\lambda B_+$. A map from the kernel to L_μ factors through its quotient by right multiplication with B_+ , and conversely any ambient map of that quotient is B -linear. Since B is generated in degree one, $B_+/B_+^2 = E \otimes U$. This proves the formula. The final assertion follows from equation (3) and $m \geq 1$. $\square$

Proof of theorem 1.1(3). Form the directed graph on isomorphism classes of simple modules, placing an arrow $L \rightarrow L'$ when $\mathrm{Ext}^1(L, L') \neq 0$. By lemma 5.2, it is the oriented Young graph for both algebras. Its unique vertex with no incoming arrow is the empty partition: every nonempty partition has a removable corner. Distance from this vertex is the number of boxes. At distance three, $(2, 1)$ is the unique vertex with two distinct immediate predecessors, namely (2) and $(1, 1)$. The other two vertices, (3) and $(1, 1, 1)$, each have one.

An equivalence of abelian categories preserves simple objects, exact sequences, vanishing of Yoneda extensions, and projective covers. It must therefore identify the two intrinsically specified covers $Q_\emptyset$ and $Q_{(2,1)}$. No choice of degree labels is supplied to the equivalence; those two labels are recovered from the graph itself. Their Hom is

$$\mathrm{Hom}_B(Q_{(2,1)}, Q_\emptyset) = \mathrm{Hom}_C(L_{(2,1)}, B_3). \quad (10)$$

For $B = A$, this vanishes by proposition 3.3. For $B = P$, the classical degree-three Cauchy decomposition identifies it with the Schur module $S_{(2,1)}U$, of dimension

$$\dim S_{(2,1)}(k^m) = \frac{m(m^2 - 1)}{3} > 0 \quad (m \geq 2).$$

An equivalence cannot turn a zero Hom group into a nonzero one. This contradiction proves the assertion, including for an equivalence not assumed k -linear. $\square$

6 Checks, prior work, and limitations

The proof above is symbolic. The accompanying programs verify the two-dimensional Hecke relations and equation (7) by exact arithmetic. A cubic relation matrix for two standard two-dimensional copies has 64 rows and 48 columns. One maximal minor is

$$\pm q^{12}(q^2 - 1)^8.$$

Thus its generic quotient has dimension $64 - 48 = 16$. At $q = 1$ the matrix instead has rank 44 and the quotient has dimension 20. This is an alternative finite-rank check of the known cubic loss.

The higher-degree checker constructs the relation matrix directly from the quadratic commutator eigenspaces, not from the character formula. Its sixteen test cases include $d = m = 2$, $q = 2$, degrees 2, 3, 4, 5 over $\mathbb{F}_{101}$, with dimensions 10, 16, 25, 36; classical controls in degrees 3, 4, 5, with

dimensions 20, 35, 56; asymmetric ranks and multiplicities; a second parameter and finite field; and $d = m = 2$, $q = 2$, degree four over the rational numbers. The same programs run in two Python versions with matching output. These are finite regression checks, not independent reviews or replacements for the general proofs.

The cubic nonflatness in [4] and the construction of [3] are prior work. Barter’s single-standard equivalence [2] is compatible with our conclusion and provides its representation-stability context. Our stated claims are the all-degree decomposition, the specified algebra fiber product, and the intrinsic non-equivalence for multiple standard copies. A bounded primary-source search is recorded in the accompanying review; it is not an exhaustive priority certificate. In particular, we make no absolute claim that no equivalent formulation exists in the literature.

AI assistance was used in mathematical exploration, drafting, and the implementation of exact checkers. No independent mathematical review or verification by a proof assistant has been completed. The author is responsible for this text. Publication as an unrefereed preprint is not certification by a journal or repository.

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