Compact-Unit Types and Fixed Vectors for Symplectic Similitudes

Alper Ferudun[*]

28 September 2026

Abstract

For every prime p and every $r \geq 1$, let $H_r = \{\mathrm{diag}(I_r, aI_r) : a \in \mathbb{Z}_p^\times\}$ inside $\mathrm{GSp}_{2r}(\mathbb{Q}_p)$. We give an elementary proof that every smooth character of H_r occurs in every irreducible infinite-dimensional smooth complex representation of this group. In particular, such a representation has a nonzero vector fixed by the subgroup congruent to H_r modulo p^m for some m. This answers the varying-level fixed-vector question recorded as Problem 3.2 of the AIM automorphic-forms problem list. The proof isolates a nontrivial character on a compact additive subgroup and enlarges that subgroup until the character stabilizer is small enough to make compact-unit averaging nonzero. One long-root subgroup and elementary symplectic transvections then suffice in every rank. The rank-two fixed-vector result and the averaging mechanism are due to Roberts–Schmidt; the argument here requires no genericity, Bessel model, or twisted-Jacquet nonvanishing theorem, and includes $p = 2$.

1 Statement and relation to the source question

Fix a prime p and an integer $r \geq 1$. We use the alternating form

$$J_r = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \qquad G_r = \mathrm{GSp}_{2r}(\mathbb{Q}_p), \qquad h(a) = \mathrm{diag}(I_r, aI_r).$$

All representations below are complex and smooth: every vector has an open stabilizer. For a smooth character $\rho : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ set

$$V[\rho] = \{v \in V : \pi(h(a))v = \rho(a)v \text{ for all } a \in \mathbb{Z}_p^\times\}.$$

Theorem 1. *Let (π, V) be an irreducible infinite-dimensional smooth representation of G_r. Then $V[\rho] \neq 0$ for every smooth character ρ of $\mathbb{Z}_p^\times$.*

Admissibility is not needed in the theorem. To express the consequence in the notation of the source question, let

$$\begin{aligned} K_r(m) &= \ker\big(\mathrm{GSp}_{2r}(\mathbb{Z}_p) \longrightarrow \mathrm{GSp}_{2r}(\mathbb{Z}/p^m\mathbb{Z})\big), \\ R_r(m) &= \{g \in \mathrm{GSp}_{2r}(\mathbb{Z}_p) : g \equiv h(a) \pmod{p^m} \text{ for some } a \in \mathbb{Z}_p^\times\}. \end{aligned}$$

Corollary 2. *Under the hypotheses of Theorem 1, there exists $m \geq 1$ such that $V^{R_r(m)} \neq 0$.*

[*]Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: AlperTheKing.

Proof. Write $H_r = h(\mathbb{Z}_p^\times)$. Direct multiplication gives $R_r(m) = H_r K_r(m)$. The theorem with $\rho = 1$ gives $0 \neq v \in V^{H_r}$. Since the $K_r(m)$ form a neighborhood basis at the identity, smoothness gives m such that this same v is fixed by $K_r(m)$. It is then fixed by $R_r(m)$. □

The question attributed to Saha in [1, Problem 3.2] uses the same letter for rank and level. We use r for the fixed rank and m for the varying level. This interpretation agrees with the separately parametrized compact subgroups in [3, Section 3.1], where the congruence is exactly to $\mathrm{diag}(I_r, aI_r)$, without off-diagonal stars. We make no claim that $m = r$ suffices.

The case $r = 2$ of Corollary 2 follows from Roberts–Schmidt [2, Corollary 7.2.4]. Their Section 7.2 uses additive-character localization and compact-unit averaging, starting with a nonzero twisted Jacquet module. We retain this averaging idea but obtain nonvanishing using only characters of compact additive groups. This makes the proof uniform in rank. The one-dimensional exception, and a limitation of the affine-group argument, are discussed in Section 4.

2 A compact-unit spectrum lemma

Put $U = (\mathbb{Q}_p, +)$ and $H = \mathbb{Z}_p^\times$. Write their elements as $u(x)$ and $h(a)$, with the semidirect-product convention

$$h(a)u(x)h(a)^{-1} = u(a^{-1}x). \tag{1}$$

For integers s and positive integers l, put $A_s = p^{-s}\mathbb{Z}_p$ and $C_l = 1 + p^l\mathbb{Z}_p$. If χ is a smooth character of a compact additive group A, its isotypic projector on a smooth module is

$$E_\chi v = \frac{1}{\mathrm{vol}(A)} \int_A \chi(x)^{-1}\pi(u(x))v\,dx. \tag{2}$$

These integrals are finite sums on each vector. Indeed, a smooth vector has open stabilizer in A, of finite index. Since A is abelian, the span of its orbit factors through a finite abelian quotient and decomposes into character spaces. In particular, the projectors in (2) are algebraic operators, with no completeness or unitarity assumption on the representation.

Lemma 3. *Let V be a smooth representation of $U \rtimes H$ in which U acts nontrivially. Every smooth character of H occurs in $V|_H$.*

Proof. There is some A_{s_0} acting nontrivially. Finite character decomposition on the orbit of a suitable vector therefore gives $0 \neq v \in V$ and a nontrivial character θ of A_{s_0} such that

$$\pi(u(x))v = \theta(x)v \qquad (x \in A_{s_0}).$$

Let the order of θ be p^c, where $c \geq 1$. Fix a smooth character ρ of H. Choose $l \geq 1$ such that C_l fixes v and ρ is trivial on C_l. Now choose $s \geq s_0$ with

$$d = c + s - s_0 \geq l.$$

Decompose v under A_s and choose a nonzero component $w = E_\chi v$. Then $\chi|_{A_{s_0}} = \theta$.

Under $A_s \cong \mathbb{Z}_p$, the subgroup A_{s_0} corresponds to $p^{s-s_0}\mathbb{Z}_p$. A character of order p^e restricts to one of order $p^{\max\{e-s+s_0,0\}}$. Since θ is nontrivial, it follows that χ has order exactly p^d. Thus, for some integer b prime to p,

$$\chi(p^{-s}z) = \exp(2\pi i b\bar{z}/p^d), \qquad \bar{z} \equiv z \pmod{p^d}.$$

Its stabilizer under multiplication by $\mathbb{Z}_p^\times$ is exactly C_d.

For $a \in C_d$, conjugation by $h(a)$ preserves A_s and χ, and so commutes with E_χ. Since $C_d \subseteq C_l$, we get $\pi(h(a))w = w$. This does not assert that C_l fixes w; only the smaller group C_d is required.

Normalize Haar measure on H to have total mass one and define

$$v_\rho = \int_H \rho(a)^{-1}\pi(h(a))w\,da.$$

Changing variables shows that $\pi(h(b))v_\rho = \rho(b)v_\rho$ for $b \in H$. By (1),

$$\pi(u(x))\pi(h(a))w = \chi(ax)\pi(h(a))w \qquad (x \in A_s).$$

Hence $E_\chi\pi(h(a))w = 0$ when $a \notin C_d$. For $a \in C_d$, both $\rho(a) = 1$ and $\pi(h(a))w = w$. Consequently

$$E_\chi v_\rho = \mathrm{vol}(C_d)w \neq 0. \tag{3}$$

This proves the assertion. □

The conductor calculation includes $p = 2$. In particular, $C_1 = \mathbb{Z}_2^\times$, as required when $d = 1$; no cyclicity assumption on $\mathbb{Z}_2^\times$ is used. The lemma never requires a vector that is an eigenvector for the whole noncompact group U.

3 Symplectic transvections and the proof in all ranks

We include the elementary generation argument to make clear that the proof does not depend on a classification of representations or on a normal-subgroup theorem. For a symplectic space W over $\mathbb{Q}_p$, write

$$T_{z,t}(x) = x + t\langle z, x\rangle z \qquad (z \in W,\ t \in \mathbb{Q}_p).$$

Expanding the alternating form shows that $T_{z,t}$ is symplectic, with inverse $T_{z,-t}$.

Lemma 4. *The transformations $T_{z,t}$ generate* $\mathrm{Sp}(W)$.

Proof. Let E be the group they generate. If $x, y \neq 0$ and $\langle y, x\rangle \neq 0$, then $T_{y-x,1/\langle y,x\rangle}$ sends x to y. If the pairing is zero, choose z outside the two proper hyperplanes $\langle z, x\rangle = 0$ and $\langle y, z\rangle = 0$, and take two such steps. The field is infinite, so such z exists. Thus E is transitive on nonzero vectors.

Fix e, f with $\langle e, f\rangle = 1$. Transvections with vector in $e^\perp$ fix e. They act transitively on $L = \{y : \langle e, y\rangle = 1\}$: for $y, z \in L$ with $\langle z, y\rangle \neq 0$, the preceding transformation has vector $z - y \in e^\perp$. Otherwise choose an intermediate $w \in L$ with $\langle w, y\rangle \neq 0$ and $\langle z, w\rangle \neq 0$. One can take $w = f + te$: the two pairings exclude at most two values of $t \in \mathbb{Q}_p$. Then take two steps, both fixing e.

For any $g \in \mathrm{Sp}(W)$, first left-multiply by an element of E so that $ge = e$, and then by an element of E fixing e so that $gf = f$. The resulting transformation preserves the symplectic complement of $\mathbb{Q}_pe + \mathbb{Q}_pf$. Induction on half the dimension, using transvections in that complement, completes the reduction to the identity. □

Proof of Theorem 1. Let $e_1, \ldots, e_r, f_1, \ldots, f_r$ be the standard symplectic basis and set

$$u(x) = I_{2r} + xE_{1,r+1} = T_{e_1,x}.$$

Suppose first that $u(\mathbb{Q}_p)$ acts trivially on V. Every nonzero vector z extends to a symplectic basis, so some element of $\mathrm{Sp}_{2r}(\mathbb{Q}_p)$ sends e_1 to z. Conjugation therefore shows that all $T_{z,t}$ act trivially. By Lemma 4, so does $\mathrm{Sp}_{2r}(\mathbb{Q}_p)$. The action then factors through the similitude quotient $G_r/\mathrm{Sp}_{2r}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times$.

Every irreducible smooth complex representation of $\mathbb{Q}_p^\times$ is one-dimensional. For completeness, choose a nonzero compact-unit character vector. Since $\mathbb{Q}_p^\times$ is abelian, its unit-character space is invariant under the whole group and hence is all of V. Extend that character to $\mathbb{Q}_p^\times$ by assigning p the value 1, and twist it off. The remaining action is a simple module over $\mathbb{C}[t, t^{-1}]$. Its maximal ideals are $(t - \alpha)$ with $\alpha \in \mathbb{C}^\times$, so it has dimension one. This contradicts the hypothesis on V.

Thus $u(\mathbb{Q}_p)$ acts nontrivially. Finally,

$$h(a)u(x)h(a)^{-1} = u(a^{-1}x).$$

The subgroup generated by $u(\mathbb{Q}_p)$ and $h(\mathbb{Z}_p^\times)$ is exactly the semidirect product of Lemma 3. Restricting V to this subgroup proves the theorem. □

4 Scope, exceptions, and checks

For a one-dimensional representation $\chi \circ \mu$, where μ is the similitude character, the only H_r-character is $\chi|_{\mathbb{Z}_p^\times}$. If χ is ramified, there is no H_r-fixed vector and therefore no $R_r(m)$-fixed vector at any level. The exclusion of one-dimensional representations is essential.

The faithful weight in (1) is also essential to the compact-unit lemma. For example, replace the action by $x \mapsto a^2 x$ and let p be odd. On $C_c^\infty(\mathbb{Q}_p^\times)$, define

$$(u(x)f)(t) = \psi(tx)f(t), \qquad (h(a)f)(t) = f(a^2 t),$$

where ψ is a nontrivial smooth additive character. This is a smooth representation with nontrivial additive action, but $h(-1)$ is the identity. It cannot contain a unit character taking the value -1 at -1. The actual symplectic-similitude cocharacter has weight -1 on the chosen root subgroup, so this obstruction is absent.

The argument proves existence, not a minimal-level formula or uniqueness of newvectors. Its relation to [2] is an all-rank elementary extension of the compact averaging proof, not a new assertion in rank two. A bounded literature search found no directly matching all-rank statement; this does not exclude folklore or certify absolute priority.

The accompanying exact-arithmetic checker tests 4,203 finite unit-character cases, the conductor and stabilizer identities, 150 root-conjugation identities, and reductions of 40 rational symplectic matrices in ranks one through five. It also tests the two obstructions above. These checks are diagnostics, not substitutes for the proofs or formal verification of the infinite statements.

This unrefereed note was prepared with AI assistance in derivation, literature search, algebraic checking, and exposition. The supporting record contains a detailed originating-researcher self-audit; no independent peer review or proof-assistant verification is claimed.

References

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[3] A. Saha, On ratios of Petersson norms for Yoshida lifts, *Forum Mathematicum* **27** (2015), 2361–2412. arXiv:1303.5246.