

A Cubic-Link Obstruction to Manifold Skeleta of Generalized Cluster Complexes

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Abstract

Let Φ be a finite root system of rank $n \geq 3$. We prove that the Fomin–Reading generalized cluster complex $\Delta^2(\Phi)$ is not the full $(n - 1)$ -skeleton of a finite polytopal PL manifold, with or without boundary, in any dimension. A face supported on negative simple roots has link $K_{3,3}$. A connected cubic nonplanar graph cannot be the full graph of a polytopal sphere or ball: in dimensions at least three, the graph of any three-dimensional cell would have minimum degree three and hence would exhaust the nonplanar graph. The type- A_3 example is the complex of partial quadrangulations of a decagon; the obstructing vertex splits it into two hexagons. This gives a negative answer to the universal manifold-skeleton suggestion in the 2005 AIM problem list and in Fomin–Reading’s Remark 4.5. It does not classify realizations for other parameters. The cluster-complex definitions, parabolic link rules, and rank-two projective-plane examples are prior work.

1 Statement and interpretation

Write $\Delta^m(\Phi)$ for the generalized cluster complex of Fomin and Reading [2], where m is a positive integer and Φ is a finite root system of rank n . It is a simplicial complex of dimension $n - 1$. The geometric part of AIM Problem 6.3 [1, p. 12] asks whether these complexes can be realized as skeleta of polytopal manifolds. Fomin and Reading [2, Remark 4.5] ask which such complexes, possibly all, occur as skeleta of polyhedral manifolds of dimension $n + m - 2$. Their rank-two examples are polygonal subdivisions of the real projective plane.

We use *full r -skeleton* to mean the subcomplex consisting of every cell of dimension at most r . A polytopal PL manifold is a finite polyhedral cell complex with convex polytopal cells and PL-manifold local links: the normal link of an interior face is a PL sphere, and that of a boundary face is a PL ball. In particular, the usual geometric face-to-face notion is included. All assertions concern the given cells, without subdivisions or additional faces in the prescribed skeleton.

Theorem 1. *Let Φ be a finite root system of rank $n \geq 3$. There is no finite polytopal PL manifold P , with or without boundary, such that*

$$\mathrm{sk}_{n-1} P \cong \Delta^2(\Phi).$$

The dimension of P is unrestricted.

Thus already $\Delta^2(A_3)$ refutes the universal realization suggestion. The obstruction in the proposed dimension $n + m - 2 = n$ is simply the nonplanarity of a local link graph. The additional argument with three-polytope cells excludes higher dimensions.

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The Cohen–Macaulay and wedge-of-spheres assertions appearing beside the geometric question in the AIM report are separate results. Their subsequent proof, including higher Cohen–Macaulay connectivity, belongs to Athanasiadis and Tzanaki [3, Theorem 1.1]. We make no new claim about those properties. Nor does Theorem 1 classify the pairs (Φ, m) which admit other manifold realizations.

2 The local graph obstruction

Lemma 2. *Let G be a connected cubic simple graph. A nonempty subgraph $H \subseteq G$ of minimum degree at least three must equal G .*

Proof. Each vertex of H must retain all three incident edges of G . Consequently all its neighbors belong to H . Connectivity propagates this conclusion to every vertex and every edge of G . $\square$

Lemma 3. *A connected cubic nonplanar graph is not the full 1-skeleton of a polytopal cell structure on a PL sphere or ball.*

Proof. There are no edges in dimension zero. A one-dimensional sphere or ball is a circle or interval, whose cell graph has degree at most two. The graph of a two-dimensional sphere or ball embeds in the sphere and is planar.

If the dimension is at least three, the cell structure contains a three-dimensional convex polytope Q as a cell; if necessary, take a three-dimensional face of a cell of larger dimension. The graph of Q is a subgraph of the prescribed graph G . Every vertex of Q has degree at least three, since its vertex figure is a polygon. Lemma 2 forces the graph of Q to equal G . But the graph of a convex three-polytope embeds in its boundary two-sphere, contradicting nonplanarity. $\square$

Only the elementary necessary graph properties of a convex three-polytope enter this proof. We do not use the converse direction of Steinitz’s theorem.

Corollary 4. *Suppose a simplicial complex K is the full r -skeleton of a polytopal PL manifold. If a face F of dimension $r - 2$ has a connected cubic link graph, that graph is planar.*

Proof. The normal face link L in the manifold is a PL sphere or ball. Its vertices and edges correspond to cofaces of F of dimensions $r - 1$ and r , respectively. All these cofaces belong to K . Thus the full 1-skeleton of L is $\text{lk}_K(F)$. Its cells are face figures of polytopes and therefore are polytopal. Apply Lemma 3. $\square$

3 A decagon counterexample

The type- A polygon model [2, Section 5.1] describes $\Delta^2(A_3)$ as the complex of partial quadrangulations of a convex decagon. Label the polygon vertices $0, \dots, 9$ in cyclic order. The vertices of the complex are the diagonals which cut the decagon into two polygons with even numbers of vertices; faces are pairwise noncrossing sets of these diagonals.

Consider the complex vertex $d = (0, 5)$. It cuts the decagon into the hexagons

$$(0, 1, 2, 3, 4, 5), \quad (0, 5, 6, 7, 8, 9).$$

The allowable diagonals compatible with d are exactly

$$\begin{aligned} A &= \{(0, 3), (1, 4), (2, 5)\}, \\ B &= \{(0, 7), (5, 8), (6, 9)\}. \end{aligned}$$

The three diagonals within either hexagon cross pairwise. Every diagonal in A is compatible with every diagonal in B . Consequently

$$\mathrm{lk}_{\Delta^2(A_3)}(d) = K_{3,3}.$$

This is a connected cubic nonplanar graph. For example, Euler's formula bounds the number of edges of a simple bipartite planar graph on six vertices by $2 \cdot 6 - 4 = 8$, whereas $K_{3,3}$ has nine. Corollary 4, with $r = 2$, proves Theorem 1 for A_3 .

The accompanying exact checker enumerates the decagon complex using two crossing predicates: cyclic endpoint interlacing and integer orientation determinants for vertices (t, t^2) . Both give

$$(f_0, f_1, f_2) = (15, 55, 55).$$

Every edge lies in exactly three triangles. There are five six-vertex links $K_{3,3}$ and ten eight-vertex cubic links. These finite checks document the example; the obstruction itself is the preceding written argument.

4 All finite root systems of rank at least three

We recall the particular established cluster-complex facts needed for the extension [2, Theorems 3.4 and 3.7, Definition 3.8, Example 4.3]. A negative simple root is compatible with a colored root precisely when the latter's simple-root expansion omits that simple root. Compatibility restricts to parabolic subsystems. The complex of a product of root systems is the join of their complexes. Finally, $\Delta^2(A_1)$ consists of three isolated vertices: the negative simple root and two colored positive roots.

Proof of Theorem 1. There are two orthogonal simple roots α_i, α_j . Indeed, normalize any three simple roots to have unit length. If each pair were nonorthogonal, their pairwise inner products would all be at most $-1/2$: a nontrivial Coxeter bond has inner product $-\cos(\pi/q)$ with $q \geq 3$. The squared norm of their sum would then be at most zero, contrary to positive definiteness and linear independence.

Let F consist of all $n - 2$ negative simple roots other than $-\alpha_i, -\alpha_j$. These are pairwise compatible, so F is a face of dimension $n - 3$. Compatibility with F forces support into the two remaining simple roots. Their subsystem is $A_1 \times A_1$. The recalled parabolic and product rules give

$$\mathrm{lk}_{\Delta^2(\Phi)}(F) = \Delta^2(A_1 \times A_1) = \Delta^2(A_1) * \Delta^2(A_1) = K_{3,3}.$$

Corollary 4, with $r = n - 1$, is a contradiction. $\square$

Irreducibility and crystallographic type are not required. The role of the cubic degree is specific to the parameter two: the corresponding rank-two product at general parameter m has graph $K_{m+1, m+1}$, to which the saturation argument above does not apply when $m > 2$.

5 Scope and prior geometric examples

A graph can be the global skeleton of a surface while being forbidden as the *local* link of a manifold face. This distinction matters here. Fomin and Reading [2, Remark 4.5] describe $\Delta^2(I_2(a))$ as the graph of a polygonal subdivision of $\mathbb{RP}^2$. Pfeifle's 2005 slides [4, slides 63–67] also discuss the octagon example using four quadrilaterals and an octagonal face. Such a global projective-plane model does not provide a sphere or ball as a local face link.

Our argument does not replace those rank-two constructions. It uses the known compatibility and parabolic-link rules to obtain a forbidden local graph in every rank at least three.

The stronger statement in arbitrary ambient dimension assumes polytopal cells. We do not assert the same conclusion for arbitrary high-dimensional CW attachments, for subdivisions, or for embeddings as an unspecified subcomplex.

This note is an unrefereed preprint. A bounded primary-source search did not locate the exact negative statement proved here; no absolute priority claim is made for an elementary consequence of established methods. AI assistance was used in developing the argument, searching sources, preparing the exact checker, and drafting the text. The author is responsible for the claims.

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