

Homotopy Types and Shelling Certificates for the H_3 Absolute Order

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Abstract

We compute the homotopy type of the absolute order of the finite Coxeter group H_3 after deleting its identity. The resulting complex is homotopy equivalent to a wedge of four circles and 872 two-spheres. Deleting the central involution as well gives a shellable wedge of 867 two-spheres. The ideals generated by classical and dual Coxeter elements are different: after deleting the identity, they are shellable wedges of 163 and 367 two-spheres, respectively. These assertions have explicit finite shelling certificates covering 4080 facets, checked by a standalone exact program that reconstructs the group and order. The link of the central involution is the disjoint union of five hexagons. It obstructs Cohen–Macaulayness and shellability for the full H_3 complex, and the same interval gives an obstruction in H_4 . The results concern these exceptional cases, not a uniform classification for finite Coxeter groups.

1 Absolute order and the main result

Let (W, S) be a finite Coxeter system, let T be its set of reflections, and let ℓ_T be reflection length. Absolute order is defined by

$$u \leq_T v \iff \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).$$

Write $\text{Abs}(W)$ for this poset and $\Delta(P)$ for the order complex of a poset P . A face of $\Delta(P)$ is a chain in P . Since the identity e is a minimum, $\Delta(\text{Abs}(W))$ is a cone. We therefore study the complex with e deleted. This is not the proper part of a single bounded noncrossing-partition interval: all maximal elements of the absolute order are retained.

Problem 3.1 of the AIM workshop collection on braid groups, clusters and free probability, attributed to V. Reiner, asks about topology and shellability of absolute order and of the ideal of parabolic Coxeter elements [1]. For the latter phrase, a distinction matters in noncrystallographic type. Fix a simple system S . Classical Coxeter elements are conjugates of products of all members of S , each used once. Dual Coxeter elements allow products of a reflection-compatible Coxeter generating set instead; see [4, Definition 2.2 and Remark 2.3]. Denote their downward ideals by J_{cl} and J_{dual} .

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Theorem 1. *Let W be of type H_3 , with identity e and central involution z . Put*

$$K = \Delta(\text{Abs}(W) \setminus \{e\}), \quad K_0 = \Delta(\text{Abs}(W) \setminus \{e, z\}).$$

Then

$$K \simeq \bigvee^4 S^1 \vee \bigvee^{872} S^2, \quad K_0 \simeq \bigvee^{867} S^2.$$

Moreover,

$$\Delta(J_{\text{cl}} \setminus \{e\}) \simeq \bigvee^{163} S^2, \quad \Delta(J_{\text{dual}} \setminus \{e\}) \simeq \bigvee^{367} S^2.$$

The three complexes other than K are shellable. The link of z in K is the disjoint union of five six-cycles. Consequently K is not Cohen–Macaulay over any field and is not shellable.

Corollary 2. *The minimum-deleted absolute-order complex of type H_4 is not Cohen–Macaulay over any field and is not shellable.*

The shellability assertions in Theorem 1 are computer-assisted finite proofs: the supplementary source archive supplies the complete ordered facet lists and a separate exact checker. The topological deductions, including the homotopy type rather than just homology, are given below. Corollary 2 does not determine the full H_4 homotopy type.

There are important precedents. Athanasiadis and Kallipoliti prove homotopy Cohen–Macaulayness of the global type- A absolute order [2]. Kallipoliti distinguishes the type- B Coxeter-generated ideal from the full order, proves shellability of closed intervals, and gives a type- D_4 obstruction [3]. Gobet treats absolute intervals below involutions [5]; his lattice classification includes the central H_3 interval, but a bounded lattice need not have connected proper part. The recent shellability theorem of Cainelli, Paolini and Zaccaria concerns individual intervals below quasi-Coxeter elements [6], not unions of such intervals. None of these distinctions can be dropped when interpreting the complexes here. We do not claim type- A or type- B EL-labelings or a solution of the uniform AIM question.

2 An exact finite model

Use the group $W = A_5 \times C_2$, with multiplication

$$(a, \epsilon)(b, \delta) = (ab, \epsilon + \delta \pmod{2}),$$

where ab applies b first. Set $e = (1, 0)$, $z = (1, 1)$ and

$$s_0 = ((23)(45), 1), \quad s_1 = ((12)(34), 1), \quad s_2 = ((25)(34), 1).$$

Each s_i is an involution, and the orders of s_0s_1, s_1s_2, s_0s_2 are 5, 3, 2. Generator closure gives 120 elements. Thus the homomorphism from the standard order-120 Coxeter presentation of type H_3 is an isomorphism. Conjugating these generators gives exactly the fifteen elements $(a, 1)$ for which a is a double transposition. These form T .

The checker constructs this group and reflection set, then computes all reflection lengths by breadth-first search. The rank distribution is

$$(1, 15, 59, 45).$$

Direct every reflection edge $g \rightarrow gt$ toward increasing length. Reachability defines the same order as the length identity: the checker verifies their equivalence on all $120^2 = 14\,400$ ordered pairs. In particular, no multiplication table, length table or comparability table is supplied as trusted input.

For $S = \{s_0, s_1, s_2\}$, the conjugacy class of $s_0s_1s_2$ has twelve elements and contains all six product orders. Its downward ideal has rank distribution $(1, 15, 47, 12)$. Enumerating the conjugates of all products of distinct members of S , including products of length zero, one and two, gives exactly the same ideal. This also checks the classical parabolic description.

For the dual convention, reflection rigidity in finite type [4, Section 2.3] ensures that every reflection-compatible simple system of this group has the H_3 diagram. Order the vertices of this diagram so that its successive labels are 5, 3 and its remaining label is 2. Enumerating triples (a, b, c) of distinct reflections with

$$\text{ord}(ab) = 5, \quad \text{ord}(bc) = 3, \quad \text{ord}(ac) = 2, \quad \langle a, b, c \rangle = W$$

gives 120 triples. Each is a Coxeter generating set, since the corresponding presentation maps onto a group of the same order, 120. Taking every product order for these triples gives 24 dual Coxeter elements. They are exactly the elements of group-theoretic order ten. Their ideal has rank distribution $(1, 15, 59, 24)$; independently enumerating all shorter products gives the same ideal. Thus the two parabolic conventions genuinely yield different complexes.

3 The central interval

Lemma 3. *The complex $L = \Delta((e, z))$ is the disjoint union of five six-cycles.*

Proof. The rank-one elements are $(a, 1)$, with a a double transposition. Each $(a, 0)$ has reflection length two: in the Klein four subgroup containing a , it is the product of the other two double transpositions, both given central bit one. It is neither the identity nor a reflection. Since $z(a, 1) = (a, 0)$, the length of z is three. The last-reflection characterization of a cover shows that its rank-two predecessors are exactly these fifteen elements $(a, 0)$.

For double transpositions a, b , the comparison $(a, 1) <_T (b, 0)$ holds exactly when $(ab, 1)$ is a reflection. Equivalently, $a \neq b$ and a, b commute. The centralizer of a double transposition in A_5 is the Klein four group on its four moved letters. Indeed, a commuting permutation preserves its unique fixed letter, and the even part of its order-eight centralizer on the other four letters is this Klein four group.

There are five such Klein four groups, one for each fixed letter. Their sets of three nonidentity elements partition the fifteen double transpositions. A fixed letter therefore gives three rank-one and three rank-two vertices, with an edge precisely when the two labels differ. This is $K_{3,3}$ with a perfect matching removed, a six-cycle. No comparison joins different fixed-letter blocks. $\square$

Since z is maximal, L is the vertex link of z in K . It has nonzero reduced H_0 over every field, while its dimension is one. The necessary Cohen–Macaulay link condition fails. The complex K is pure of dimension two; purity is checked by extending every vertex and edge to a maximal chain. A pure shellable complex is Cohen–Macaulay, so K cannot be shellable. This proves the last assertion of Theorem 1.

4 Complete shelling certificates

Encode a vertex by the lexicographic position, numbered 0 through 119, of (p, ϵ) , where $p = (p(0), \dots, p(4))$ is a zero-based permutation tuple. This fixes the interpretation of every facet

independently of the order in which a program generates the group.

The supplementary files `shelling-search.json` and `dual-shelling-search.json` contain ordered triangle lists. The first contains certificates for $J_{\text{cl}} \setminus \{e\}$ and for K_0 ; the second contains the certificate for $J_{\text{dual}} \setminus \{e\}$. The checker `verify_h3.py` reads only these lists as certificate data. It reconstructs every expected facet from its own absolute order, verifies equality of the listed and expected facet sets with no repetitions, and checks that each vertex and edge lies in a triangle.

Lemma 4. *Let $F_1, \dots, F_N$ be all the distinct triangular facets of a pure two-dimensional simplicial complex. Suppose that for every $k > 1$, the edges of F_k already present in earlier facets form a nonempty set, and their endpoints are exactly the vertices of F_k already present. Then the list is a shelling. If b of its triangles have all three edges already present, the complex is homotopy equivalent to $\bigvee^b S^2$.*

Proof. The intersection of F_k with the preceding complex contains no triangle. The stated condition says exactly that this intersection is nonempty and pure of dimension one, which is the shelling condition in dimension two. If it consists of one edge or two edges, adjoining F_k preserves homotopy type; the new simplex collapses onto that boundary subcomplex. If it contains all three edges, a two-cell is attached along a loop. Inductively the preceding complex is a wedge of two-spheres, starting with a contractible triangle, so this loop is nullhomotopic. The attachment then adds one two-sphere. $\square$

For every facet the checker implements this test by maintaining the sets of old vertices and old edges. Its complete results are displayed in Table 1. The final three columns exclude the first triangle; their sum is one less than the number of facets. There are $600 + 1200 + 2280 = 4080$ certified facets in total.

Complex	f_0	f_1	f_2	1 edge	2 edges	3 edges
$\Delta(J_{\text{cl}} \setminus \{e\})$	74	510	600	71	365	163
$\Delta(J_{\text{dual}} \setminus \{e\})$	98	930	1200	95	737	367
K_0	118	1530	2280	115	1297	867

Table 1: Complete shellings and numbers of triangles attached along each possible number of old edges.

Lemma 4 now proves the three shellability and wedge-of-two-spheres assertions. The sphere count also equals $f_0 - f_1 + f_2 - 1$ in each case. Boundary-matrix calculations used in exploration are not needed for the proof: finite-field homology alone would not establish these homotopy equivalences or exclude integral torsion.

5 Restoring the central vertex

Every chain containing z consists of z and a chain strictly below it. Consequently

$$K = K_0 \cup (z * L), \quad K_0 \cap (z * L) = L.$$

The checker verifies this decomposition face by face. The complex K_0 is connected and simply connected, since it is a wedge of 867 two-spheres. Each of the five circle components of L has a nullhomotopic inclusion in K_0 . After contracting these loops, choose paths from their image points to a common point. This makes the entire inclusion $L \rightarrow K_0$ homotopic to one constant map.

A simplicial inclusion is a cofibration, and its mapping cone is unchanged up to homotopy when the attaching map is replaced by a homotopic map. Hence

$$K \simeq K_0 \vee \Sigma L,$$

where Σ is the *unreduced* suspension. This convention is essential because L is disconnected. The suspension of five disjoint circles can be viewed as five two-spheres with all north poles identified and all south poles identified. A compatible CW structure has two vertices, five meridian edges and five two-cells. Each two-cell has a nullhomotopic attaching loop on its own meridian. Contracting one meridian as a maximal tree leaves four loop edges and five two-spheres. Thus

$$\Sigma L \simeq \bigvee^4 S^1 \vee \bigvee^5 S^2, \quad K \simeq \bigvee^4 S^1 \vee \bigvee^{872} S^2.$$

This proves the remaining homotopy assertion. The full complex has f -vector $(119, 1560, 2310)$, so its reduced Euler characteristic is $868 = 872 - 4$, consistently with the result.

6 The inherited obstruction in type H4

Proof of Corollary 2. Choose a standard parabolic subgroup of type H_3 in H_4 , and let z be its central involution. Absolute intervals below an element of a finite parabolic subgroup agree with those computed inside it. Indeed, if $x \leq_T z$, the fixed-space description of absolute order gives inclusion of moved spaces and hence inclusion of parabolic closures $P(x) \subseteq P(z)$. Here $P(z)$ is precisely the chosen H_3 subgroup. Reflection length agrees in that subgroup and the ambient group by the fixed-space formula; see also the finite-parabolic discussion in [7].

The moved space of z is the three-dimensional span of the H_3 roots. Let s be a simple reflection outside this parabolic subgroup. The three roots in a reduced reflection factorization of z are linearly independent. Adding the root of s preserves independence. Carter's reflection-length criterion gives $\ell_T(zs) = 4$.

In the minimum-deleted H_4 order complex, consider the edge with vertices z, zs . There is no rank between their ranks three and four, and no element has rank greater than four. Its link is therefore exactly $\Delta((e, z))$, the disjoint union of five hexagons. The ambient complex is pure of dimension three: in a finite rank-four Coxeter group any shorter reduced reflection factorization can be extended by a root outside its moved space, and every chain extends to a maximal chain. Its one-dimensional edge link has nonzero reduced H_0 over every field. This violates the Cohen–Macaulay link condition and rules out shellability. $\square$

7 Reproducibility and scope

Extract the accompanying source archive and run the following commands in its `reproducibility` directory:

```
python3 -B verify_h3.py
python3 -O -B verify_h3.py
```

The program uses only the Python standard library and exact finite operations. It contains no `assert` statements, so optimization does not remove a test. Both runs produce the same JSON report, including SHA-256 hashes of the checker and certificates. The report covers all 14 400 order comparisons, all three complete shellings, their purity and the cone decomposition. Eighteen deliberately corrupted certificates are rejected; these include correct facet sets in orders whose next triangle meets the preceding complex in no edge or only in isolated vertices. The checker does not import the exploratory or certificate-search programs and does not trust their success flags.

The certificates establish finite statements about the specified group model; the written arguments translate them to the homotopy assertions. This is not proof-assistant verification or independent expert review. The bounded literature review, current through 20 October 2026, is supplied with the source package; it is not an exhaustive novelty or priority certification. The broad AIM problem remains only partially addressed. AI tools assisted the author with exploration, drafting and exact-checker development; the manuscript is self-audited and unrefereed.

References

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