

Geometric Sampling Does Not Determine Regularized R -Diagonal Laws

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Abstract

For every $s_0 > 0$, $q > 1$ and finite integer $K \geq 1$, we construct a continuum of distinct affiliated R -diagonal laws having the same joint star-moments and the same first K alternating cumulants of $c(1 + sc^*c)^{-1/2}$ at every scale $s = s_0 q^k$, $k \in \mathbb{Z}$. All absolute moments are finite. A finite combination of classical lognormal Stieltjes perturbations gives an explicit resolvent difference with prescribed finite-order zeros on the entire grid, and an exact formula identifies a higher cumulant separating every pair. Conversely, a Blaschke–Jensen argument gives a sufficient condition under which samples of the first regularized cumulant determine the law without moment assumptions. The result is a discrete reconstruction obstruction, not a new construction of the established radial analytic R -transform.

1 Regularized data and the reconstruction question

Let (M, τ) be a finite von Neumann algebra with faithful normal normalized trace. An affiliated R -diagonal operator has the distribution of $c = UH$, where $H \geq 0$ is affiliated, U is a Haar unitary and U is free from H ; see [2, 4]. Put $X = H^2$ and let σ be its probability law on $[0, \infty)$. Its bounded regularization is

$$b_s = c(1 + sc^*c)^{-1/2}, \quad s > 0, \quad \|b_s\| \leq s^{-1/2}. \quad (1)$$

It is again R -diagonal, with squared modulus $X/(1 + sX)$. Write $\alpha_j(s)$ for its alternating free cumulant of length $2j$. Traciality makes the two alternating orientations equal. Define

$$\begin{aligned} L_\sigma(s) &= \int_0^\infty \frac{1}{1 + sx} d\sigma(x), \\ a_\sigma(s) &= \int_0^\infty \frac{x}{1 + sx} d\sigma(x) = \frac{1 - L_\sigma(s)}{s}, \\ m_j(s) &= \int_0^\infty \left(\frac{x}{1 + sx} \right)^j d\sigma(x). \end{aligned} \quad (2)$$

In particular $\alpha_1(s) = a_\sigma(s)$.

The full function a_σ determines σ by Stieltjes uniqueness. This elementary encoding also appears in the inherited discussion of the unbounded-pair question in [8]. It is natural to ask how much information remains when only a finite number of cumulants is retained at each of a discrete sequence of scales. Here is an explicit obstruction.

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Theorem 1. Fix $s_0 > 0$, $q > 1$ and an integer $K \geq 1$. There are pairwise distinct affiliated R -diagonal distributions c_θ , $-1 < \theta < 1$, such that:

1. every absolute moment of every c_θ is finite, and all their joint star-moments, hence their formal joint R -series, agree;
2. for every $k \in \mathbb{Z}$ and $1 \leq j \leq K$,

$$\alpha_{j,\theta}(s_0 q^k) = \alpha_{j,0}(s_0 q^k);$$

3. their full radial analytic R -transforms differ. At each sampled scale, an explicitly specified higher alternating cumulant separates every two distributions in the family.

The number K may be arbitrarily large but is fixed for each family. The theorem does not assert equality of all cumulants at a positive scale. The moment indeterminacy of lognormal distributions and their sine perturbations are classical; see [5, 6]. The construction here prescribes a finite multiplicity of zeros of the transform difference on an arbitrary two-sided geometric grid.

The AIM Free Analysis list [1, p. 7] asks for an R -transform of an unbounded R -diagonal pair. The affiliated framework and the analytic transform of the symmetrized modulus are established [3, 4]. Our theorem concerns a discrete reconstruction scheme motivated by that question; it does not claim that AIM proposed this scheme or that its entire open-ended question is newly resolved. No absolute priority for the explicit construction is asserted.

2 Recovering cumulants from resolvent derivatives

The alternating cumulants of b_s are the even scalar free cumulants of the symmetrized law of $|b_s|$. To see this, expand the alternating word of length $2j$ by noncrossing partitions. Only even alternating blocks survive. In any noncrossing partition all of whose blocks have even size, the gaps between successive vertices of a block are unions of even-sized blocks. Each gap therefore has even size, so the vertices of the block alternate in parity. These are precisely the same terms as in the even moment expansion for the symmetrized modulus. Thus

$$\alpha_j(s) = m_j(s) + P_j(m_1(s), \dots, m_{j-1}(s)), \quad (3)$$

where P_j is a universal polynomial. For example,

$$\alpha_1 = m_1, \quad \alpha_2 = m_2 - 2m_1^2, \quad \alpha_3 = m_3 - 6m_1m_2 + 7m_1^3.$$

Differentiating (2) gives

$$m_j(s) = \frac{(-1)^{j-1}}{(j-1)!} a_\sigma^{(j-1)}(s). \quad (4)$$

Differentiation under the integral is justified locally uniformly for $s > 0$, since $x^j/(1+sx)^j$ is bounded uniformly in $x \geq 0$ when s stays in a compact positive interval. No undamped moments are needed.

Consequently an N -fold zero of $L_\sigma - L_\rho$ at a positive scale forces equality of the first N regularized cumulants there. At order $N+1$, the cumulant difference equals the moment difference because all lower moments already agree.

3 Finite harmonic synthesis of the counterexample

Let $N \geq K$ be the least odd integer, write $N = 2R - 1$, and put

$$v = \log q, \quad \mu = \frac{v}{2} - \log s_0. \quad (5)$$

For $0 \leq j < R$, define

$$\begin{aligned} p_j &= 2^{-(N-1)} \binom{N}{R-1-j}, \\ B_j &= p_j \exp\left(-\frac{(2j+1)^2 \pi^2}{2v}\right), \quad B_R = 0, \\ b_r &= \exp\left(\frac{2\pi^2 r^2}{v}\right) (B_{r-1} - B_r) \quad (1 \leq r \leq R), \quad B = \sum_{r=1}^R b_r. \end{aligned} \quad (6)$$

The positive coefficients p_j , hence B_j , strictly decrease with j . Thus $b_r > 0$ and $B > 0$. Set

$$h(y) = \frac{1}{B} \sum_{r=1}^R b_r \sin\left(\frac{2\pi r(y - \mu)}{v}\right). \quad (7)$$

Then $|h(y)| \leq 1$, and h is not identically zero. Give Y_θ the density

$$\frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{(y - \mu)^2}{2v}\right) [1 + \theta h(y)], \quad -1 < \theta < 1, \quad (8)$$

and set $X_\theta = e^{Y_\theta}$ and $H_\theta = \sqrt{X_\theta}$.

3.1 Normalization and equality of all moments

If Y is the unperturbed Gaussian, completing the square gives, for every integer ℓ and every positive integer r ,

$$\mathbb{E} \left[e^{\ell Y} \sin\left(\frac{2\pi r(Y - \mu)}{v}\right) \right] = e^{\ell\mu + v\ell^2/2 - 2\pi^2 r^2/v} \sin(2\pi r\ell) = 0. \quad (9)$$

The case $\ell = 0$ normalizes (8). Its positivity follows from $|\theta| < 1$. All integer moments of X_θ , including negative ones, therefore equal $\exp(\ell\mu + v\ell^2/2)$. Every real absolute moment is finite, since the density is at most twice the Gaussian density. Different values of θ give different laws of X_θ .

Realize $c_\theta = UH_\theta$ in the tracial free product of the commutative probability space carrying X_θ and a Haar-unitary probability space. The positive multiplication operator H_θ is densely defined and affiliated. All mixed moments of c_θ, c_θ^* exist by noncommutative Hölder, and their values agree by the R -diagonal moment-cumulant relations. The affiliated distributions nevertheless differ because their modulus-square laws differ. No free infinite divisibility assumption or conclusion is made.

3.2 The exact resolvent difference

Lemma 2. For $t = \log s + \mu$, define

$$D_r(s) = \mathbb{E} \left[\frac{\sin(2\pi r(Y - \mu)/v)}{1 + se^Y} \right].$$

Then

$$D_r(s) = -\sqrt{\frac{2\pi}{v}} e^{-t^2/(2v)} \sum_{j=0}^{r-1} e^{-2\pi^2 r^2/v + (2j+1)^2 \pi^2/(2v)} \cos\left(\frac{(2j+1)\pi t}{v}\right). \quad (10)$$

Proof. Before taking imaginary parts, the defining integral is

$$\frac{1}{\sqrt{2\pi v}} \int_{\mathbb{R}} \frac{e^{-u^2/(2v) + 2\pi i r u/v}}{1 + e^{t+u}} du.$$

Completing the square and substituting $z = u - 2\pi i r$ changes this to $e^{-2\pi^2 r^2/v}/\sqrt{2\pi v}$ times the integral, from left to right on $\text{Im } z = -2\pi r$, of

$$f(z) = \frac{e^{-z^2/(2v)}}{1 + e^{t+z}}.$$

The denominator is unchanged by the integer imaginary period. The lower line minus the real line encloses the strip counterclockwise. The poles are $z_j = -t - (2j+1)\pi i$, $0 \leq j < r$, and $\text{Res}_{z_j} f = -e^{-z_j^2/(2v)}$. Gaussian decay makes the vertical sides vanish. The real-line integral is real, so taking imaginary parts of the residue theorem gives (10). There are no poles on either horizontal boundary. $\square$

Write $L_\theta = L_{\sigma_\theta}$, where σ_θ is the law of X_θ . The coefficients in (6) telescope:

$$e^{(2j+1)^2 \pi^2/(2v)} \sum_{r=j+1}^R b_r e^{-2\pi^2 r^2/v} = e^{(2j+1)^2 \pi^2/(2v)} B_j = p_j.$$

Since $\cos^N x = \sum_{j=0}^{R-1} p_j \cos((2j+1)x)$, the lemma gives the exact identity

$$L_\theta(s) - L_0(s) = -\theta C e^{-(\log s + \mu)^2/(2v)} \cos^N\left(\frac{\pi(\log s + \mu)}{v}\right), \quad C = \frac{\sqrt{2\pi/v}}{B} > 0. \quad (11)$$

Completion of the proof of Theorem 1. At $s_k = s_0 q^k$, (5) gives $\log s_k + \mu = v(k+1/2)$. For $\theta \neq 0$, the right side of (11) has a zero of exactly order N there. Multiplication by $-1/s$ preserves that order, so $a_\theta - a_0$ and its first $N-1$ derivatives vanish at s_k . Equations (4) and (3) give equality of $\alpha_1, \dots, \alpha_N$, and hence of the required first K .

At the next order, the lower moments still coincide. Differentiating the leading term at the zero, using that N is odd, gives

$$\alpha_{N+1,\theta}(s_k) - \alpha_{N+1,0}(s_k) = \theta(-1)^k C \left(\frac{\pi}{v}\right)^N \frac{e^{-v(k+1/2)^2/2}}{s_k^{N+1}}. \quad (12)$$

Replacing θ by $\theta - \eta$ separates any two family members. The right side is nonzero for $\theta \neq \eta$ at every finite k .

For an affiliated R -diagonal c , the symmetrized modulus law ν_c determines its distribution. Its scalar analytic transform $\mathcal{R}_c = R_{\nu_c}$ is information-complete: on a suitable reciprocal Stolz domain it satisfies $R_{\nu_c}(w) = G_{\nu_c}^{(-1)}(w) - 1/w$, where $G_\nu(z) = \int (z-x)^{-1} d\nu(x)$ [3, 4]. Equality of these transforms would imply equality of their Cauchy transforms and hence of the modulus laws, contrary to the construction. This proves the remaining assertion. $\square$

Remark 3. When K is even, $N = K+1$: the construction matches one more cumulant than required and first separates at order $K+2$. The variance $v = \log q$ and the unperturbed lognormal law are independent of K ; only the finite harmonic perturbation changes.

4 A sufficient condition for uniqueness of samples

The geometric obstruction contrasts with the following elementary application of the zero-set condition for bounded holomorphic functions.

Proposition 4. *Let s_j be distinct positive numbers. If*

$$\sum_j \frac{\sqrt{s_j}}{1 + s_j} = \infty, \quad (13)$$

then the values $a_\sigma(s_j)$ determine every probability measure σ on $[0, \infty)$, without moment assumptions.

Proof. For $\operatorname{Re} z > 0$ put $F_\sigma(z) = zL_\sigma(z^2)$. Its integrand is holomorphic and locally uniformly bounded in $x \geq 0$: on compact sets of z , the denominator has no zero for bounded x , and $z/(1+xz^2)$ tends uniformly to zero as $x \rightarrow \infty$. Moreover

$$\operatorname{Re} \frac{z}{1+xz^2} = \frac{(\operatorname{Re} z)(1+x|z|^2)}{|1+xz^2|^2} > 0.$$

Thus $H_\sigma = (F_\sigma - 1)/(F_\sigma + 1)$ maps the right half-plane holomorphically into the unit disk. If two a -functions agree at s_j , the bounded holomorphic difference $H_\sigma - H_\rho$ has zeros $z_j = \sqrt{s_j}$. The Cayley map takes them to $w_j = (z_j - 1)/(z_j + 1)$ in the disk.

A nonzero bounded holomorphic function on the disk has $\sum_j (1 - |w_j|) < \infty$. Indeed, remove a possible finite-order zero at the origin and apply Jensen's formula; letting the radius tend to one bounds $\sum_j -\log |w_j|$, which dominates the stated sum. For $z > 0$,

$$\frac{z}{1+z^2} \leq 1 - \left| \frac{z-1}{z+1} \right| \leq \frac{2z}{1+z^2}.$$

Condition (13) therefore forces $H_\sigma = H_\rho$. It follows that $L_\sigma = L_\rho$, and Stieltjes uniqueness gives $\sigma = \rho$. $\square$

The sequences $s_j = j^2$ and $s_j = j^{-2}$, $j \geq 1$, satisfy (13). For a two-sided geometric grid the sum is finite; Theorem 1 supplies explicit nonuniqueness even after all undamped moments and any fixed finite number of regularized cumulants are supplied. We do not claim that (13) is necessary for every sampling sequence, or claim a new general theorem about rational approximation with prescribed poles.

At one fixed positive scale, the *entire* bounded joint R -series does determine the original R -diagonal law. Its moments determine the compactly supported law of $X/(1+sX)$, which has no atom at $1/s$; the inverse map $y \mapsto y/(1-sy)$ recovers the law of X . This does not contradict the fixed finite-order sampling theorem.

5 Scope and reproducibility

The radial analytic transform and the free-convolution identity for symmetrized moduli are prior work [3, 4]. Recent work treats freely infinitely divisible R -diagonal elements [7]; the present family is not asserted to belong to that subclass. The lognormal moment-null perturbations are also classical [5, 6]. The point of this note is the explicit finite-harmonic synthesis (6)–(11) and its sampled-cumulant consequence, with the separating formula (12).

The general argument is analytic. Accompanying exact rational and Laurent polynomial regressions check the coefficient synthesis, multiplicity and noncrossing moment-cumulant identities. Independent high-precision quadrature checks the contour formula and the first separating moment in selected cases. These finite and floating-point checks are not substitutes for the proof, and the numerical errors are not interval-certified.

This unrefereed note was developed with AI assistance in derivation, source comparison, computation and exposition. It has been self-audited, but independent review and proof-assistant verification have not occurred. It makes no claim of absolute priority, a universal analytic multi-variable R -transform, or closure of the full AIM question.

References

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