

Fourier Shadows and Tracial Free Infinite Divisibility

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Abstract

We compare tensor and free addition of bounded tracial tuples whose joint laws are freely infinitely divisible, after projecting both laws to the scalar series $\tau(e^{i\langle u, X \rangle})$. The Fourier shadows obtainable from free sums form a proper subset of products of such shadows in every dimension at least two. A witness is the product of two variance-one semicircle Fourier transforms on separate coordinate axes. Every tracial joint lift of this product fails to have a free fifth root: a fixed sum of Hermitian squares has formal expectation $-2/25$. We also give a commutator obstruction to free roots of commuting tuples, and an explicit pair of distinct bounded tracial free compound Poisson laws with identical Fourier shadows. These results address a specified bounded-tracial comparison, not a classification of all noncommutative series or a nontracial extension of the AIM free-analysis question.

1 The comparison and its scope

For a bounded selfadjoint k -tuple X in a C^* -probability space $(\mathcal{A}, \tau)$, its joint law records all ordered moments. We call

$$\Phi_X(u) = \tau \left(\exp \left(i \sum_{j=1}^k u_j X_j \right) \right), \quad u \in \mathbb{R}^k,$$

its *Fourier shadow*. Boundedness makes the scalar power series entire. Its coefficients are symmetrized ordered moments; in general it does not determine the joint law. For noncommuting coordinates we do not assert that this shadow is a positive-definite function on $\mathbb{R}^k$.

The first question in section 0.10 of the AIM free-analysis problem list [1], attributed to Nica, asks how the scalar Fourier series from tensor addition of jointly freely infinitely divisible tuples relate to the noncommutative series from free addition. Here we compare both operations after the same scalar projection, and explicitly require a tracial state. The source paragraph does not explicitly impose traciality. The bounded-tuple framework and its Boolean-to-free Bercovici–Pata bijection are treated in [2]; that bijection does not identify the two classes studied here.

Write $\mathcal{F}_k$ for the Fourier shadows of bounded tracial tuples whose joint laws are freely infinitely divisible, and put

$$\mathcal{T}_k = \{fg : f, g \in \mathcal{F}_k\}.$$

Let σ be the centered variance-one semicircle law and let $f(u) = \int e^{iux} d\sigma(x)$.

Theorem 1.1. *For every $k \geq 2$ one has $\mathcal{F}_k \subsetneq \mathcal{T}_k$. The function $f(u_1)f(u_2)$ belongs to $\mathcal{T}_k$ but not to $\mathcal{F}_k$. More strongly, any tracial joint law with this Fourier shadow has no positive formal free power of exponent $0 < t < 1/4$, and hence has no free n -th root for any integer $n \geq 5$.*

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Free sums of input laws in the defining class remain in that class. Conversely, every such law is a free sum with the zero tuple. Thus $\mathcal{F}_k$ is exactly the set of shadows obtainable from free sums of two such input laws. Products of shadows are obtained by placing the two tuples in tensor-independent factors. The strict inclusion therefore compares the two operations in common scalar coordinates. It does not classify either set, address arbitrary nontracial states, or settle the scalar case $k = 1$.

2 A lift-independent sum-of-squares certificate

We use the standard noncrossing moment–cumulant relation

$$\mu(X_{i_1} \cdots X_{i_n}) = \sum_{\pi \in \text{NC}(n)} \prod_{V \in \pi} \kappa_{i_V}. \quad (1)$$

For $t > 0$, let μ_t be the unital formal law obtained by replacing each nonempty cumulant by $t\kappa$. It need not be positive. Any free n -th root, if one exists, must have law $\mu_{1/n}$, since free cumulants linearize addition. Cumulants of a tracial law are cyclic: this follows inductively from (1), rotating the noncrossing partitions along with the word. Scaling preserves this cyclicity.

Proof of Theorem 1.1. Consider a tracial lift of $f(u)f(v)$, and denote its first two coordinates by X, Y . Its Fourier coefficients prescribe zero means, covariance I_2 , pure fourth moments 2, and a sum of mixed fourth moments equal to 6. Cyclicity leaves two mixed fourth moments:

$$A = \mu(X^2Y^2), \quad B = \mu(XYXY), \quad 4A + 2B = 6.$$

The four cyclic rotations of $XXYY$ have moment A ; the two alternating words have moment B . Put $a = \kappa_{XXYY} = A - 1$ and $b = \kappa_{XYXY} = B$. The covariance pairing contributions are 1 and 0, respectively, so

$$4a + 2b = 2. \quad (2)$$

Pure fourth cumulants vanish. For the formal power,

$$\mu_t(X^4) = \mu_t(Y^4) = 2t^2, \quad \mu_t(X^2Y^2) = ta + t^2, \quad \mu_t(XYXY) = tb.$$

Consequently

$$\begin{aligned} \mu_t((X^2 - Y^2)^2) &= 2t^2 - 2ta, \\ \mu_t([X, Y]^*[X, Y]) &= 2t^2 + 2t(a - b). \end{aligned}$$

For the sum of Hermitian squares

$$S = 3(X^2 - Y^2)^2 + [X, Y]^*[X, Y] \quad (3)$$

we obtain, independently of the unknown lift,

$$\mu_t(S) = 8t^2 - 4ta - 2tb = 2t(4t - 1). \quad (4)$$

This is negative for $0 < t < 1/4$, and equals $-2/25$ at $t = 1/5$. It rules out positivity, including for any hypothetical root not initially presented in a tracial realization: its cumulants would still force (4).

Finally, the zero tuple gives $1 \in \mathcal{F}_k$, so $\mathcal{F}_k \subseteq \mathcal{T}_k$. The two axis tuples $(s, 0, \dots, 0)$ and $(0, r, 0, \dots, 0)$, with s, r variance-one semicircles, are jointly freely infinitely divisible by their scalar semicircle roots. Their tensor sum has shadow $f(u_1)f(u_2)$, which the preceding argument excludes from $\mathcal{F}_k$. Extra zero coordinates give the result for every $k \geq 2$. $\square$

The argument proves only a necessary bound on t . It does not assert that $t = 1/4$ is achievable for some lift.

3 Commutators and commuting free roots

Proposition 3.1. *For a bounded selfadjoint pair, center the variables and set $a = \phi(X^2)$, $b = \phi(Y^2)$, $c = \phi(XY)$, and $C = \phi([X, Y]^*[X, Y])$. No trace is required for the identity*

$$C(\mu_t) = tC + 2t(t-1)(ab - \operatorname{Re}(c^2)). \quad (5)$$

For a tracial jointly freely infinitely divisible pair, $C \geq 2 \det \operatorname{Cov}(X, Y)$.

Proof. The noncrossing-pair contributions to $XY YX$, $XY XY$, $YXYX$, and $YXXY$ are, in order,

$$ab + |c|^2, \quad c^2 + |c|^2, \quad \bar{c}^2 + |c|^2, \quad ab + |c|^2.$$

Their signed sum in the commutator square is $Q = 2ab - c^2 - \bar{c}^2$. The fourth-cumulant contribution scales by t , while this pairing contribution scales by t^2 , giving $t(C - Q) + t^2Q$. Centering changes neither commutators nor cumulants of degree at least two, so the calculation also covers nonzero means. In the tracial case c is real. Apply positivity at $t = 1/n$, divide by t , and let n tend to infinity. $\square$

Corollary 3.2. *If a bounded commuting tuple has a free n -th root for some $n \geq 2$, its joint spectral measure is supported on an affine line, or is a point mass. A bounded commuting joint law is freely infinitely divisible if and only if it is an affine-line pushforward of a scalar freely infinitely divisible law, including the deterministic case.*

Proof. For each coordinate pair the commutator square is zero. At $t = 1/n$, (5) and positivity force its covariance determinant to vanish. A real positive-semidefinite matrix whose pairwise principal minors all vanish has rank at most one: all vectors in a Gram representation are pairwise dependent. Every linear functional orthogonal to the covariance range has zero variance under the joint spectral measure. A finite basis of these functionals confines the measure to an affine line, or a point.

In the nondegenerate case write the law as $m + vZ$, with v real and nonzero and Z centered. Projecting any joint free root along v gives the corresponding scalar root. Conversely, a scalar n -th root Z_n gives the joint root $m/n + vZ_n$, by multilinearity of cumulants and the drift rule. No commuting assumption was imposed on a hypothetical joint root. $\square$

For the actual commuting pair of independent variance-one semicircles, (5) gives $C(\mu_t) = 2t(t-1)$. This particular lift has no free n -th root for any $n \geq 2$, a stronger obstruction than the lift-independent one in Theorem 1.1.

More generally, sum corresponding coordinates of r tensor-independent free variance-one semicircular pairs. Their covariance is rI_2 . Their commutator is the sum of the centered factor commutators, which are tensor-orthogonal and each have squared norm 2. Thus $C = 2r$, and (5) becomes

$$C(\mu_t) = 2rt(rt - r + 1).$$

For $t > 0$, positivity requires $t \geq 1 - 1/r$. No sufficiency for this bound is claimed.

4 Conditional positivity and Fourier nonuniqueness

The following elementary consequence records the role of traciality. Let $K(1) = 0$ and let K agree with the free cumulants on nonempty words. For a jointly freely infinitely divisible law and a polynomial p with zero constant term,

$$K(p^*p) = \lim_{n \rightarrow \infty} n\mu_{1/n}(p^*p) \geq 0. \quad (6)$$

This is the standard conditional-positivity argument underlying the Fock-space theory in [3, Theorem 4.7.1].

Lemma 4.1. *A bounded tracial jointly freely infinitely divisible law whose pure fourth cumulants all vanish is jointly semicircular, with its allowed means and covariance.*

Proof. The seminorm defined by (6) makes every X_i^2 null. Cauchy–Schwarz gives $K(X_i^2 X_j^2) = 0$. Cyclicity then makes $K((X_i X_j)^*(X_i X_j)) = 0$. Every degree-two word is therefore null. Every word of length at least three is p^*q with p of degree two and q of positive degree. Another Cauchy–Schwarz application shows that its cumulant vanishes. $\square$

Even inside the bounded tracial FID class, scalar Fourier information does not recover the joint law. The next example uses standard free compound Poisson machinery; no novelty for that machinery is claimed.

Proposition 4.2. *There exist two different bounded tracial jointly freely infinitely divisible laws of three selfadjoint variables with the same Fourier shadow.*

Proof. Let

$$a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b_j = a_j^T,$$

and let tr_2 be normalized matrix trace. Fix $\lambda > 0$. Use the compound Poisson cumulants

$$\kappa_{i_1 \dots i_n} = \lambda \text{tr}_2(a_{i_1} \cdots a_{i_n}), \tag{7}$$

and the analogous cumulants with b .

For completeness, a bounded realization takes $H = L^2(M_2, \text{tr}_2)$, $\xi_j = \sqrt{\lambda} a_j$, and the selfadjoint left multiplication $T_j h = a_j h$. On full Fock space put

$$Z_j = \ell(\xi_j) + \ell(\xi_j)^* + \Lambda(T_j),$$

where ℓ is left creation and $\Lambda(T)$ applies T to the first tensor factor and kills the vacuum. This is the scalar specialization of the standard gauge construction [3, sections 4.6–4.7]. The operators are selfadjoint and have norm at most $2\sqrt{\lambda} + 1$. In the vacuum expansion, a noncrossing block of size at least two contributes

$$\langle \xi_{i_1}, T_{i_2} \cdots T_{i_{n-1}} \xi_{i_n} \rangle = \lambda \text{tr}_2(a_{i_1} \cdots a_{i_n});$$

inner excursions give the nested blocks. Singleton contributions are zero, as are the traces of the a_j . This proves (7). Replacing λ by λ/n constructs all free roots. Cyclic cumulants give a tracial law; its tracial GNS representation provides a von Neumann realization. The same holds for b .

For real $u \in \mathbb{R}^3$, the Pauli identity gives

$$(u_1 a_1 + u_2 a_2 + u_3 a_3)^2 = \|u\|^2 I,$$

and transposition gives the same identity for b . Both linear combinations have trace zero, so all their matrix moments agree. Formula (7) now gives identical free cumulants, and hence identical bounded laws, for every scalar linear combination of the two compound Poisson tuples. Their Fourier shadows agree. But their ordered third moments are $i\lambda$ and $-i\lambda$, since $a_1 a_2 a_3 = iI$ and transposition reverses the product inside the trace. Thus the joint laws differ. $\square$

5 Verification and limits

The accompanying standard-library Python checker uses Gaussian rational arithmetic, noncrossing partition inversion, finite matrix states, and direct tensor and gauge-Fock actions. Its 2,973 assertions passed in Python 3.13 and 3.12; the replay is not counted as an additional suite. A separate Node checker uses symbolic $\mathbb{Z}[t, a]$ arithmetic to verify (4) and exact negative-root certificates, passing 97 assertions. These finite controls supplement the proofs, rather than proving the general statements by enumeration.

The bounded-tracial set comparison is the main scope of this note. The nontracial lifting problem, a full classification of series and the one-dimensional comparison are not settled. The elementary corollaries may be familiar to specialists. Standard cumulant and Fock-space results are credited, and a bounded literature search does not establish absolute novelty or priority. This is an unrefereed preprint. AI assistance was used in developing the arguments, source analysis, exact checks and manuscript preparation; no independent human review or proof-assistant verification is claimed.

References

- [1] American Institute of Mathematics, *Free Analysis: Problems*, 24 August 2006, section 0.10, p. 7. <https://aimath.org/WWN/freeanalysis/freeanalysis.pdf>.
- [2] S. T. Belinschi and A. Nica, *η -series and a Boolean Bercovici–Pata bijection for bounded k -tuples*, *Advances in Mathematics* **217** (2008), 1–41. [arXiv:math/0608622](https://arxiv.org/abs/math/0608622).
- [3] R. Speicher, *Combinatorial theory of the free product with amalgamation and operator-valued free probability theory*, *Memoirs of the American Mathematical Society* **132** (1998), no. 627. [doi:10.1090/memo/0627](https://doi.org/10.1090/memo/0627).