A Slice–Riesz–Bochner Characterization of Joint Brown Determinant Functions

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Abstract

We characterize the functions arising as Fuglede–Kadison determinant transforms of bounded commuting tuples in a fixed II_1 factor. After recovering the log-modulus formula intended by an AIM range question, we show that such functions are exactly the logarithmic potentials of compact probability measures on $\mathbb{C}^n$. Intrinsically, each complex line must give a normalized subharmonic logarithmic potential with uniformly bounded Riesz support, and the unit-frequency transforms of the slice measures must form a continuous positive-definite function on $\mathbb{C}^n$. Complex scaling of all slice measures is forced automatically by the single global function; Bochner's theorem then reconstructs the unique joint measure. Every compact probability measure is realized by a commuting normal tuple in any prescribed II_1 factor. In one variable, the representing measure is simply the Riesz measure of one transformed function.

1 The range question and its formula

The 2006 AIM *Free Analysis* list asks which functions on $\mathbb{C}^n$ have the integral representation in the immediately preceding joint-Brown-measure display [1]. The printed display omits absolute-value bars on its right side. Read literally, it cannot equal the real quantity $\log \Delta(X) = \tau(\log |X|)$ and has no global complex-logarithm branch. Schultz's final theorem and its explicit final-primary restatement fix the intended identity as [6, 2]

$$\log \Delta\left(1 - \sum_{j=1}^{n} \alpha_j T_j\right) = \int_{\mathbb{C}^n} \log \left| 1 - \sum_{j=1}^{n} \alpha_j \zeta_j \right| \, d\mu_T(\zeta). \tag{1}$$

Here $T_1, \ldots, T_n$ commute in a II_1 factor (M, τ) and μ_T is their compactly supported joint Brown probability measure.

Write

$$a \cdot \zeta = \sum_{j=1}^{n} a_j \zeta_j$$

for the complex bilinear pairing, and let $B_R \subseteq \mathbb{C}^n$ be the closed Euclidean ball.

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Theorem 1.1 (complete range criterion)**.** *Fix a* II_1 *factor* (M,τ)*, and let* $F:\mathbb{C}^n\to[-\infty,\infty)$*. The following are equivalent.*

1. *There are commuting* $T_1,\dots,T_n\in M$ *such that*

$$F(\alpha)=\log\Delta\left(1-\sum_j \alpha_j T_j\right).$$

2. *There are* $R<\infty$ *and a Borel probability measure* μ *supported in* B_R *such that*

$$F(\alpha)=\int_{\mathbb{C}^n}\log|1-\alpha\cdot\zeta|\,d\mu(\zeta).$$

3. *We have* $F(0)=0$*, and there is* $R<\infty$ *such that, for every* $a\in\mathbb{C}^n$*, the function*

$$U_a(\lambda)=\log|\lambda|+F(a/\lambda),\qquad \lambda\neq 0,$$

admits a subharmonic extension to $\mathbb{C}$*, not identically* $-\infty$*, with*

$$U_a(\lambda)-\log|\lambda|\longrightarrow 0\quad(|\lambda|\to\infty).$$

Its Riesz measure

$$\nu_a=\frac{1}{2\pi}\Delta U_a$$

has support in $\{|w|\le R\|a\|_2\}$*, and*

$$\Phi(a)=\int_{\mathbb{C}} e^{i\,\mathrm{Re}\,w}\,d\nu_a(w)$$

is continuous and positive definite on the additive real group $\mathbb{C}^n\simeq\mathbb{R}^{2n}$*.*

The Riesz measures in item 3 are automatically probabilities. When these conditions hold, the representing measure is unique and

$$\nu_a=(a\cdot)_*\mu\qquad(a\in\mathbb{C}^n).$$

The operators in item 1 may be chosen normal in the originally fixed factor M*.*

Positive definiteness means

$$\sum_{j,k=1}^{N} c_j\overline{c_k}\,\Phi(a_j-a_k)\ge 0$$

for every finite choice of $a_j\in\mathbb{C}^n$ and $c_j\in\mathbb{C}$.

2 A planar normalization lemma

We normalize the distributional Laplacian by

$$\frac{1}{2\pi}\Delta_\lambda \log|\lambda - w| = \delta_w.$$

Lemma 2.1. *Let u be subharmonic on $\mathbb{C}$, not identically $-\infty$. Suppose its Riesz measure $\nu = (2\pi)^{-1}\Delta u$ has compact support and*

$$u(\lambda) - \log|\lambda| \longrightarrow 0 \quad (|\lambda| \to \infty).$$

Then $\nu(\mathbb{C}) = 1$ and

$$u(\lambda) = \int_{\mathbb{C}} \log|\lambda - w|\, d\nu(w)$$

as locally integrable functions and in the upper-semicontinuous representatives.

Proof. Put $m = \nu(\mathbb{C})$ and $p(\lambda) = \int \log|\lambda - w|\, d\nu(w)$. Distributionally, $h = u - p$ is an entire harmonic function. If the support lies in $|w| \leq S$, then uniformly on large circles,

$$p(\lambda) = m\log|\lambda| + \int \log|1 - w/\lambda|\, d\nu(w) = m\log|\lambda| + o(1).$$

Thus $h(\lambda) = (1-m)\log|\lambda| + o(1)$. Taking circular means and using the mean-value property gives

$$h(0) = (1-m)\log r + o(1),$$

so $m = 1$. Now $h \to 0$ at infinity; harmonic Liouville makes $h = 0$. $\square$

In particular, the extension in item 3 of Theorem 1.1 is unique. Two candidates agree off 0, hence their Riesz measures agree there; both have mass one, so their atoms at 0 agree as well.

3 Necessity of the slice conditions

Assume Theorem 1.1(2). For $a \in \mathbb{C}^n$ and $\lambda \neq 0$,

$$\begin{aligned} U_a(\lambda) &= \log|\lambda| + \int \log|1 - (a\cdot\zeta)/\lambda|\, d\mu(\zeta) \\ &= \int \log|\lambda - a\cdot\zeta|\, d\mu(\zeta). \end{aligned}$$

This supplies the subharmonic extension and gives

$$\nu_a = (a\cdot)_*\mu. \tag{2}$$

If μ is supported in B_R, then $|a\cdot\zeta| \leq R\|a\|_2$, proving the support condition and the normalized growth at infinity.

Finally,

$$\Phi(a) = \int_{\mathbb{C}^n} e^{i\,\mathrm{Re}(a\cdot\zeta)}\, d\mu(\zeta) \tag{3}$$

is the characteristic function of μ for the real duality pairing $(a,\zeta) \mapsto \mathrm{Re}(a\cdot\zeta)$. This pairing parametrizes every real character: if $a = x + iy$ and $\zeta = u + iv$, its value is $x\cdot u - y\cdot v$. Thus Φ is continuous and positive definite.

4 Automatic scaling and Bochner reconstruction

Assume Theorem 1.1(3). Lemma 2.1 makes every ν_a a probability and gives

$$U_a(\lambda) = \int \log|\lambda - w|\, d\nu_a(w).$$

For $s \neq 0$, the definition of U_a gives away from zero

$$U_{sa}(\lambda) = \log|s| + U_a(\lambda/s).$$

The Riesz measures of the two sides agree off zero. Their difference is supported at zero and has total mass zero, hence vanishes. Therefore

$$\nu_{sa} = s_*\nu_a \qquad (s \in \mathbb{C}). \tag{4}$$

The case $s = 0$ follows from $U_0(\lambda) = \log|\lambda|$ and $\nu_0 = \delta_0$.

Because $\Phi(0) = 1$, Bochner's theorem on $\mathbb{R}^{2n}$ [5] supplies a unique probability measure μ on $\mathbb{C}^n$ with characteristic function (3). Fix $a \in \mathbb{C}^n$. At every $s \in \mathbb{C}$, the characteristic function of $(a\cdot)_*\mu$ is

$$\Phi(sa).$$

By the definition of Φ and (4), this equals

$$\int_{\mathbb{C}} e^{i\,\mathrm{Re}(sw)}\, d\nu_a(w).$$

Uniqueness of characteristic functions on $\mathbb{R}^2$ proves

$$(a\cdot)_*\mu = \nu_a \qquad (a \in \mathbb{C}^n). \tag{5}$$

5 Sharp support and recovery of the function

Let D be a countable dense subset of the unit sphere in $\mathbb{C}^n$. For each $a \in D$, the support hypothesis and (5) give $|a \cdot \zeta| \leq R$ almost surely. Intersect the countably many full-measure events and extend the inequality to every unit a by continuity. Since for $\zeta \neq 0$

$$\sup_{\|a\|_2=1} |a \cdot \zeta| = \|\zeta\|_2, \qquad a = \overline{\zeta}/\|\zeta\|_2,$$

we obtain $\mu(B_R) = 1$. The countable dense set avoids an illegitimate uncountable intersection of almost-sure events.

Lemma 2.1 and (5) now give

$$U_a(\lambda) = \int_{\mathbb{C}^n} \log|\lambda - a\cdot\zeta|\, d\mu(\zeta).$$

At $\lambda = 1$, the left side is $F(a)$, proving Theorem 1.1(2). The characteristic function determines μ, so the representing measure is unique.

These statements are understood distributionally at logarithmic poles. The positive part is finite for compact positive measures; the value may be $-\infty$ but never $+\infty$, and no subtraction of pointwise infinities is used.

6 Realization in every fixed factor

Schultz's theorem gives (2) from (1). Conversely, let μ be any compactly supported probability measure on $\mathbb{C}^n$. Every II_1 factor contains a diffuse abelian von Neumann subalgebra: recursively split the identity into equal-trace dyadic projections and take the weak closure. It contains a trace copy of $L^\infty([0,1])$.

There is a bounded measurable map $Z = (Z_1, \dots, Z_n) : [0,1] \to \mathbb{C}^n$ with law μ. Regard its coordinate functions as bounded commuting normal elements $T_j \in M$. Abelian functional calculus gives

$$\begin{aligned} \log \Delta\left(1 - \sum_j \alpha_j T_j\right) &= \tau\left(\log\left|1 - \sum_j \alpha_j T_j\right|\right) \\ &= \int_{\mathbb{C}^n} \log|1 - \alpha \cdot \zeta| \, d\mu(\zeta). \end{aligned}$$

Schultz's uniqueness identifies their joint Brown measure with μ. This proves (1) from (2) in the originally prescribed factor and completes Theorem 1.1.

7 The one-variable form

Corollary 7.1. *Let $F : \mathbb{C} \to [-\infty, \infty)$ with $F(0) = 0$. It has a representation*

$$F(\alpha) = \int_{|\zeta| \le R} \log|1 - \alpha\zeta| \, d\mu(\zeta)$$

by a probability measure if and only if

$$U(\lambda) = \log|\lambda| + F(1/\lambda)$$

extends subharmonically to $\mathbb{C}$, has normalized logarithmic growth, and has Riesz measure supported in $\overline{D(0,R)}$. Then

$$\mu = \frac{1}{2\pi} \Delta U.$$

Proof. Necessity is the slice calculation above. Conversely, Lemma 2.1 reconstructs the probability measure; evaluating its potential at $1/\alpha$ and adding $\log|\alpha|$ recovers F. □

8 Why local holomorphic data do not suffice

For $\|\alpha\|_2 < 1/R$, one may use the principal branch

$$G_\mu(\alpha) = \int \mathrm{Log}(1 - \alpha \cdot \zeta) \, d\mu(\zeta),$$

whose derivatives record holomorphic moments. Such moments do not determine a positive measure, as is familiar in multivariable Fantappiè problems [4, 3]. In one variable, normalized arclength σ_r on $|\zeta| = r$ satisfies Jensen's identity

$$\int \log|1 - \alpha\zeta| \, d\sigma_r(\zeta) = \log^+(r|\alpha|).$$

Thus σ_r and δ_0 give the same function on $|\alpha| < 1/r$ but different global functions. The global slice Laplacians retain the radial information that local holomorphic moments lose.

9 Scope

The theorem characterizes the bounded global log-modulus problem fixed by the AIM context and final joint-Brown-measure literature. It makes no range claim for the source's literally misprinted complex logarithm, signed or complex measures, unbounded affiliated operators, or branch-dependent holomorphic representations on general zero-free domains.

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