

Symmetric Unimodality for Real Free Convolution Powers

Alper Ferudun*

11 October 2026

Abstract

We derive preservation of symmetric weak unimodality under every real free additive convolution power $t \geq 1$, without support, moment or density assumptions. The essential input is the Cauchy-transform inequality in Anonymous's unrefered preprint *Symmetric unimodality under free additive convolution* (2026). We prove an exact transport identity for its inequality defect along power subordination. Consequences include preservation under free compression and an interval description of the unimodal times of a symmetric free Lévy process. Combined with an example of Hasebe and Sakuma, this gives a symmetric freely infinitely divisible law whose powers are unimodal exactly for $t \geq 1$. This is a dependent continuation of the cited transform inequality, not a new proof of the original pairwise symmetric-unimodality conjecture.

1 Statement and dependence on the transform inequality

A probability measure is *weakly unimodal about c* if it has the form $p\delta_c + f(x)dx$, where $p \geq 0$, f is nondecreasing to the left of c , and nonincreasing to its right. We consider measures also symmetric about c . A central atom, an unbounded density at the center, and unbounded support are allowed.

For a Borel probability μ on $\mathbb{R}$, write

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(u)}{z - u}, \quad F_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbb{C}^+.$$

For real $t \geq 1$, the free power $\mu^{\boxplus t}$ exists. For $t > 1$, its subordination map $\omega_t : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ satisfies

$$G_{\mu^{\boxplus t}}(z) = G_\mu(\omega_t(z)), \quad t\omega_t(z) = z + (t-1)F_{\mu^{\boxplus t}}(z). \quad (1)$$

These are established facts, valid without moment assumptions; see [6, Section 2, equations (2.4)–(2.5)] and the references therein. The laws obey $(\mu^{\boxplus s})^{\boxplus t} = \mu^{\boxplus st}$ for $s, t \geq 1$. For a freely infinitely divisible law the same identity holds for all positive parameters, and the resulting free Lévy semi-group is weakly continuous [4, Section 2.2].

Put $Q = \{x + iy : x > 0, y > 0\}$. Whenever $G'_\mu \neq 0$, define

$$\mathcal{D}_\mu(z) = \operatorname{Im} \frac{1}{G'_\mu(z)} + 2 \operatorname{Im}(z) \operatorname{Re} \frac{1}{G_\mu(z)}. \quad (2)$$

The substantive input to this note is

$$\mathcal{D}_\mu(z) \leq 0 \quad (z \in Q) \quad (3)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

for finite mixtures of centered uniform laws, proved in [1, Proposition 1]. That source uses an analytic moment-extremal argument and establishes the pairwise closure theorem posed in the AIM Free Analysis list [2] and [5, Conjecture 3.5]. Its main theorem and inequality are not contributions of the present note.

The cited release is explicitly unrefereed and AI-assisted. Our local audit accepts its written argument; neither its publication nor this use asserts community acceptance or formal verification. Theorems below are consequences of (3) and established subordination. The exact transport identity is proved here, rather than inferred from the integer-power case.

Theorem 1.1. *Let μ be a symmetric weakly unimodal probability about c . For every real $t \geq 1$, the law $\mu^{\boxplus t}$ is symmetric and weakly unimodal about tc . For $c = 0$, $t > 1$ and $z \in Q$, both derivatives in the following identity are nonzero, and*

$$\mathcal{D}_{\mu^{\boxplus t}}(z) = t \mathcal{D}_{\mu}(\omega_t(z)). \quad (4)$$

Integer powers follow from pairwise closure. Passing to arbitrary real powers is not justified by rational approximation alone: a general probability need not have the requisite free convolution roots. Identity (4) treats all $t > 1$ directly. The conclusion is weak unimodality; it does not assert global smoothness at support edges or strict decrease of the unsmoothed density.

2 Uniform mixtures and the power calculation

Let U_a be the uniform law on $[-a, a]$ for $a > 0$, and set $U_0 = \delta_0$. The centered symmetric weakly unimodal probabilities are precisely

$$\mu = \int_{[0, \infty)} U_a d\rho(a) \quad (5)$$

with ρ a probability measure; this is the symmetric form of the uniform-mixture characterization of unimodality. For an even nonincreasing density f , the positive-radius mixing measure is $2a(-df(a))$, and the atom at radius zero is the central atom. Tonelli's theorem verifies both the representation and its total mass, including unbounded f .

Lemma 2.1. *If μ is as in (5), then on Q*

$$\operatorname{Re} G_{\mu} > 0, \quad \operatorname{Im} G_{\mu} < 0, \quad \operatorname{Im} G'_{\mu} > 0, \quad \mathcal{D}_{\mu} \leq 0.$$

The sign of $\operatorname{Re} G_{\mu}(x + iy)$, for $y > 0$, is the sign of x .

Proof. For $a > 0$, direct integration gives

$$\operatorname{Re} G_{U_a}(x + iy) = \frac{1}{4a} \log \frac{(x + a)^2 + y^2}{(x - a)^2 + y^2} > 0, \quad G'_{U_a}(z) = -\frac{1}{z^2 - a^2}.$$

Consequently

$$\operatorname{Im} G'_{U_a}(x + iy) = \frac{2xy}{(x^2 - y^2 - a^2)^2 + 4x^2y^2} > 0.$$

The same signs hold at $a = 0$, where $G_{U_0} = 1/z$. Integration preserves strictness; differentiation under the probability integral is allowed by the bound $1/y^2$ for the Cauchy derivative kernel. Reflection proves the last assertion.

Approximate ρ weakly by finitely supported probabilities. The associated laws μ_n converge weakly to μ , since $a \mapsto U_a$ is weakly continuous, including at zero. The kernels $(z - u)^{-1}$ and $-(z - u)^{-2}$ are bounded and continuous for real u , so $G_{\mu_n} \rightarrow G_\mu$ and $G'_{\mu_n} \rightarrow G'_\mu$ at fixed $z \in \mathbb{C}^+$. The signs already proved make the limiting reciprocals legitimate on Q . Inequality (3), with its attribution above, therefore passes to the limit. $\square$

Lemma 2.2. *The centered symmetric weakly unimodal class is weakly closed.*

Proof. Suppose $\mu_n \Rightarrow \mu$ and represent each μ_n by ρ_n in (5). For $R > 0$,

$$\rho_n((2R, \infty)) \leq 2\mu_n(\{|u| > R\}),$$

because $U_a(\{|u| > R\}) \geq 1/2$ when $a > 2R$. Weak convergence makes (μ_n) tight, hence (ρ_n) is tight. A weakly convergent subsequence of ρ_n has a probability limit ρ . Weak continuity of the uniform kernel then gives $\mu = \int U_a d\rho(a)$. $\square$

Proof of Theorem 1.1. First center μ at zero and fix $t > 1$. Write $\lambda = \mu^{\boxplus t}$, $g = G_\lambda$, $F = 1/g$, and $\omega = \omega_t$. Reflection commutes with free powers, by uniqueness of the Voiculescu transform under multiplication by t ; thus λ is symmetric.

For $z \in Q$, $\operatorname{Re} g(z)$ cannot be zero. If it were, the sign assertion of Lemma 2.1 and $g = G_\mu(\omega)$ would imply $\operatorname{Re} \omega = 0$. Also $\operatorname{Re} F = 0$. Taking real parts in (1) would give $0 = \operatorname{Re} z > 0$. The nontangential asymptotic $g(z) \sim 1/z$, along the diagonal at infinity, and connectedness of Q show that $\operatorname{Re} g > 0$ throughout Q . Hence $\operatorname{Re} F > 0$ and $\omega(z) \in Q$.

Put $a = G'_\mu(\omega(z)) \neq 0$. Differentiating (1) gives

$$g' = a\omega', \quad t\omega' = 1 - (t-1)\frac{g'}{g^2},$$

and therefore

$$g' \left(\frac{t}{a} + \frac{t-1}{g^2} \right) = 1. \tag{6}$$

In particular $g' \neq 0$. Set $y = \operatorname{Im} z$ and $v = \operatorname{Im} \omega(z)$. Since $tv = y + (t-1)\operatorname{Im} F$,

$$\begin{aligned} \mathcal{D}_\lambda(z) &= t \operatorname{Im}(1/a) + (t-1) \operatorname{Im}(F^2) + 2y \operatorname{Re} F \\ &= t \operatorname{Im}(1/a) + 2((t-1) \operatorname{Im} F + y) \operatorname{Re} F \\ &= t(\operatorname{Im}(1/a) + 2v \operatorname{Re} F) = t \mathcal{D}_\mu(\omega(z)). \end{aligned}$$

This proves (4), including its nonvanishing conditions. Lemma 2.1 now yields

$$\operatorname{Im}(1/g') \leq -2y \operatorname{Re} F < 0, \quad \text{so} \quad \operatorname{Im} g' > 0 \quad \text{on } Q.$$

For every $y > 0$, the function $-\operatorname{Im} g(x + iy)/\pi$ is the density of the ordinary convolution of λ with the centered Cauchy law of scale y . It is even, and its derivative is strictly negative for $x > 0$. These smoothings are centered unimodal and converge weakly to λ as $y \downarrow 0$. Lemma 2.2 proves weak unimodality of λ .

The case $t = 1$ is immediate. Finally, translation by c before taking a free power translates the result by tc ; this follows from the same Voiculescu-transform identity. $\square$

3 Compression, onset times, and the sharp range

Corollary 3.1. *Let X be a bounded selfadjoint element of a tracial noncommutative probability space whose law is symmetric weakly unimodal about c . Let p be a projection of trace $\alpha \in (0, 1]$, free from X . The law of pXp in pAp , equipped with normalized trace $\tau(\cdot)/\alpha$, is symmetric weakly unimodal about c .*

Proof. The free-compression formula identifies this law with the pushforward of $\mu^{\boxplus 1/\alpha}$ under $u \mapsto \alpha u$; see [7, Proposition 1.4], following Nica and Speicher. Apply Theorem 1.1. This concerns the limiting/free law, not the shape of a finite empirical eigenvalue histogram. $\square$

Proposition 3.2. *For a symmetric freely infinitely divisible law η centered at zero, let*

$$I_\eta = \{s > 0 : \eta^{\boxplus s} \text{ is weakly unimodal}\}.$$

Then I_η is either empty, $(0, \infty)$, or $[s_, \infty)$ for some $s_* > 0$.*

Proof. If $s \in I_\eta$ and $u \geq s$, apply Theorem 1.1 to $(\eta^{\boxplus s})^{\boxplus u/s} = \eta^{\boxplus u}$. Thus I_η is upward closed. Weak continuity of the free Lévy semigroup, followed by Lemma 2.2, makes it relatively closed in $(0, \infty)$. These two properties give exactly the three alternatives. $\square$

Connected support alone does not imply eventual unimodality: Anshelevich and Arizmendi provide laws with support $\mathbb{R}$ none of whose free powers are unimodal [3, Lemma 14 and Corollary 15]. This separate obstruction is prior work, not a counterexample claimed here.

Corollary 3.3. *There exists a symmetric freely infinitely divisible probability λ such that*

$$\lambda^{\boxplus t} \text{ is weakly unimodal} \iff t \geq 1, \quad t > 0.$$

In particular, the range in Theorem 1.1 cannot be extended to all positive parameters, even when all roots exist.

Proof. Let C be the standard centered Cauchy law and take $\eta = \frac{1}{2}\delta_0 + \frac{1}{2}C$. This law is symmetric and weakly unimodal. Hasebe and Sakuma show that it is freely infinitely divisible but not freely s -selfdecomposable [4, Remark 4.2 and Example 4.8]. Their Theorem 4.1 implies that not all positive powers of η are unimodal.

Since $1 \in I_\eta$ and $I_\eta \neq (0, \infty)$, Proposition 3.2 gives $I_\eta = [s_*, \infty)$ for some $0 < s_* \leq 1$. Set $\lambda = \eta^{\boxplus s_*}$. For every $t > 0$, its t -th power is $\eta^{\boxplus ts_*}$, which is unimodal exactly when $ts_* \geq s_*$. $\square$

Corollary 3.3 is an existence construction via the onset time; it does not compute s_* or an explicit density for λ . Hasebe–Sakuma’s characterization concerns unimodality at *every* positive time, and already shows why one unimodal marginal is insufficient for that property. Our additional use is the upward-closed, endpoint-attaining description furnished by the real-power preservation result.

The contribution is the power transfer and these stated consequences. A bounded literature search did not locate the all-real-power statement in this generality; no exhaustive priority claim is made. AI tools assisted the derivation, source comparison, algebraic checks and preparation of this self-audited, unrefereed note. Finite exact checks support the displayed algebra but do not formally verify the analytic argument or its cited input.

References

- [1] Anonymous, *Symmetric unimodality under free additive convolution*, Evidence Press, version 0.1.0-candidate, 7 September 2026. Unrefereed AI-assisted preprint.
[doi:10.5281/zenodo.22649756](https://doi.org/10.5281/zenodo.22649756).
- [2] American Institute of Mathematics, *Free Analysis workshop problem list*, 2006, Section 0.5.
<https://aimath.org/WWN/freeanalysis/freeanalysis.pdf>.
- [3] M. Anshelevich and O. Arizmendi, *The exponential map in non-commutative probability*, International Mathematics Research Notices 2017 (17), 5302–5342. [doi:10.1093/imrn/rnw164](https://doi.org/10.1093/imrn/rnw164).
Version consulted: [arXiv:1601.08180v4](https://arxiv.org/abs/1601.08180v4).
- [4] T. Hasebe and N. Sakuma, *Unimodality for free Lévy processes*, Annales de l’Institut Henri Poincaré, Probabilités et Statistiques 53 (2) (2017), 916–936. [doi:10.1214/16-AIHP742](https://doi.org/10.1214/16-AIHP742).
Version consulted: [arXiv:1508.01285v2](https://arxiv.org/abs/1508.01285v2).
- [5] T. Hasebe and Y. Ueda, *Large time unimodality for classical and free Brownian motions with initial distributions*, ALEA 15 (2018), 353–374. [doi:10.30757/ALEA.v15-15](https://doi.org/10.30757/ALEA.v15-15).
- [6] H.-W. Huang, *Supports of measures in a free additive convolution semigroup*, International Mathematics Research Notices 2015 (12), 4269–4292. Version consulted: [arXiv:1205.5542v1](https://arxiv.org/abs/1205.5542v1).
- [7] D. Shlyakhtenko and T. Tao, with an appendix by D. Jekel, *Fractional free convolution powers*, Indiana University Mathematics Journal 71 (6) (2022), 2551–2594.
[doi:10.1512/iumj.2022.71.9163](https://doi.org/10.1512/iumj.2022.71.9163). Version consulted: [arXiv:2009.01882v3](https://arxiv.org/abs/2009.01882v3).