

Sharp Norms of Rectangular Majorana Matrices and a Two-Matrix Obstruction to Strong Convergence

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14 October 2026

Abstract

Let $C_{m,n}$ be an m by n matrix whose entries are distinct self-adjoint Clifford generators, normalized to have square one. We prove that $\|C_{m,n}\|^2 = 2(m+n-2)$ for $m, n \geq 2$; the one-row and one-column cases have squared norm $\max(m, n)$. The proof couples the vector and spin representations of $\mathfrak{so}(n)$ and identifies an attaining highest-weight summand. The square case gives the optimal uniform bound 2 for the normalized arrays in the explicit CAR subquestion of the AIM Free Analysis problem list. Several arrays of distinct generators converge jointly in star moments to a standard free circular family. Nevertheless, the rectangular formula gives an exact sum-of-squares norm that excludes strong convergence already for two arrays. The broader q-Gaussian approximation question is not addressed. We distinguish this unsymmetrized ensemble from the Hermitian fermionic ensemble studied by Shlyakhtenko.

1 The array problem and the main results

The Free Analysis problem list [1, Section 0.3, p. 3] asks whether the matrices $N^{-1/2}[c_{ij}]_{i,j=1}^N$, formed from unitary CAR generators, are uniformly bounded. We use the normalized Majorana interpretation: all entries are distinct self-adjoint unitaries and

$$c_{ij}c_{kl} + c_{kl}c_{ij} = 2\delta_{ik}\delta_{jl}1. \quad (1)$$

Ordinary annihilation operators, repeated entries, and the constraint $c_{ij}^* = c_{ji}$ define different models. In particular, the matrix with entries (1) is generally not self-adjoint.

Write Cl_K for the complex Clifford C^* -algebra on K normalized self-adjoint generators. Its canonical trace is the coefficient of the identity in the ordered-monomial basis. We compute matrix norms in a faithful representation; one choice is left multiplication on Cl_K with its canonical tracial inner product. This convention includes odd K , when the algebra has two simple summands.

Theorem 1. *Let $C_{m,n} = [c_{ij}]$ have the mn distinct entries in (1). Then*

$$\|C_{m,n}\|^2 = \begin{cases} \max(m, n), & \min(m, n) = 1, \\ 2(m+n-2), & m, n \geq 2. \end{cases} \quad (2)$$

The formula is unchanged by arbitrary signs and permutations of the distinct generators.

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Corollary 2. For $N \geq 2$,

$$\|N^{-1/2}C_{N,N}\| = 2\sqrt{1 - \frac{1}{N}}.$$

At $N = 1$ the norm is 1. Thus the optimal uniform bound is 2.

For a fixed positive integer d , take dN^2 distinct generators $c_{ij}^{(r)}$ and set

$$X_N^{(r)} = N^{-1/2}[c_{ij}^{(r)}], \quad \varphi_N = N^{-1} \text{Tr} \otimes \tau_N, \quad 1 \leq r \leq d,$$

where τ_N is the canonical Clifford trace. All generators, including those of different arrays, satisfy (1). A standard free circular family is normalized by $\varphi(x_r x_r^*) = 1$.

Theorem 3. The tuple $(X_N^{(1)}, \dots, X_N^{(d)})$ converges jointly in star moments to a standard free circular family $(x_1, \dots, x_d)$. For $N \geq 2$,

$$\left\| \sum_{r=1}^d X_N^{(r)} (X_N^{(r)})^* \right\| = 2(d+1) - \frac{4}{N}. \quad (3)$$

In the standard full-Fock realization of the limit,

$$\left\| \sum_{r=1}^d x_r x_r^* \right\| \leq (1 + \sqrt{d})^2. \quad (4)$$

Consequently the tuples do not converge strongly when $d \geq 2$.

Here strong convergence requires convergence of every noncommutative star-polynomial norm in addition to star moments. For $d = 2$, the polynomial $Z_1 Z_1^* + Z_2 Z_2^*$ has finite-array norm tending to 6, whereas its limit norm is at most $3 + 2\sqrt{2} < 6$. Equality in (4) is not needed.

Scope and prior work

The square formula and a vector-spin Casimir approach already appear in the research aid attached to record AIM-PROBABILITY-0113 of the frozen UnsolvedMath data [4]. We do not claim that square calculation as a new discovery. We give a complete rectangular argument, including the type B boundary terms, and use it to obtain the explicit two-array obstruction in Theorem 3. The representation-theoretic inputs are classical; we use the minuscule tensor rule in [2, Corollary 30.7].

Shlyakhtenko [3] proves exact norms and failure of strong convergence for fermionic matrix models using Casimir methods. His Hermitian ensemble has entries with $d_{pq}^* = d_{qp}$ and a trace-pairing Clifford form, and its normalized norm is $\sqrt{4 - 3/N}$. It is not the unsymmetrized array of individually self-adjoint, distinct entries considered here, whose normalized square norm is $\sqrt{4 - 4/N}$ for $N \geq 2$. Thus neither Casimir methods nor the general failure of strong convergence for fermionic ensembles is claimed to originate here. The two-array statement concerns precisely our ensemble, not an improvement of the matrix count for a different model in [3].

Theorem 1 answers the explicit CAR-array subquestion of [1]. It does not answer the preceding broader request for uniformly bounded approximations to arbitrary q-Gaussian families. A bounded primary-source review found no matching rectangular identity, but this is not a certificate of priority.

2 Column rotations and a Casimir identity

Take $S = \text{Cl}_{mn}$ with its faithful left regular action. For $1 \leq a < b \leq n$, put

$$L_{ab} = E_{ab} - E_{ba}, \quad R_{ab} = \frac{1}{2} \sum_{i=1}^m c_{ia} c_{ib}.$$

Extend these definitions by skew-symmetry and zero diagonal. The anticommutation relations give

$$[R_{ab}, c_{ik}] = \delta_{bk} c_{ia} - \delta_{ak} c_{ib}$$

and hence

$$[R_{ab}, R_{cd}] = \delta_{bc} R_{ad} - \delta_{ac} R_{bd} - \delta_{bd} R_{ac} + \delta_{ad} R_{bc}.$$

Thus $L_{ab} \mapsto R_{ab}$ is a representation of $\mathfrak{so}(n)$. Both L_{ab} and R_{ab} are skew-adjoint.

For a representation T define the positive Casimir $\Omega_T = -\sum_{a < b} T(L_{ab})^2$. On the vector representation $W = \mathbb{C}^n$, it equals $(n-1)I$. Expanding the matrix product gives

$$\begin{aligned} C_{m,n}^* C_{m,n} &= mI + 2 \sum_{a < b} L_{ab} \otimes R_{ab} \\ &= mI + \Omega_W \otimes 1 + 1 \otimes \Omega_S - \Omega_{W \otimes S}. \end{aligned} \tag{5}$$

The sign in this identity follows by squaring $L_{ab} \otimes 1 + 1 \otimes R_{ab}$. We use the Gram operator, not $C_{m,n}^2$.

3 Spin weights and vector tensor products

Assume $n \geq 3$ and write $\ell = \lfloor n/2 \rfloor$. In orthonormal weight coordinates the half-sum of positive roots and the Casimir eigenvalue are

$$\rho_j = \frac{n-2j}{2}, \quad q(\lambda) = \sum_{j=1}^{\ell} \lambda_j (\lambda_j + n - 2j). \tag{6}$$

The normalization agrees with $q(e_1) = n-1$ and follows from the usual highest-weight expression for the Casimir. Dominance means $\lambda_1 \geq \dots \geq \lambda_{\ell} \geq 0$ in type B, and $\lambda_1 \geq \dots \geq \lambda_{\ell-1} \geq |\lambda_{\ell}|$ in type D.

Let Δ_n be the spin module for odd n , and the direct sum of the two half-spin modules for even n . Its weights are all sign vectors $\frac{1}{2}(\pm 1, \dots, \pm 1)$, each once. Each spin constituent is minuscule. The minuscule tensor rule says that tensoring V_{λ} with such a constituent gives each $V_{\lambda+\nu}$ for which $\lambda + \nu$ is dominant, once; see [2, Corollary 30.7 and Sections 30.4, 31.2–31.3].

Lemma 4. *Under column rotations, S is a positive integer multiple of $\Delta_n^{\otimes m}$. In particular, every highest weight λ of S satisfies $|\lambda_j| \leq m/2$.*

Proof. Order Clifford monomials by rows. As a vector space, S is the tensor product of the m row Clifford algebras. Left multiplication by an even element of one row commutes past earlier rows without a sign. The Lie action is therefore the tensor sum of the row actions; no ungraded tensor-product claim for odd Clifford algebras is needed.

In one row, the Cartan operators $-\frac{i}{2} c_{i,2j-1} c_{i,2j}$ have all sign weights $\pm \frac{1}{2}$ with equal positive multiplicity. Indeed, in any joint spectral projector, only the constant monomial has nonzero canonical trace. The row character is consequently a positive integer multiple of the character of Δ_n . Complete reducibility proves the tensor-power assertion. All weights of that tensor power have coordinates between $-m/2$ and $m/2$, including its highest weights. $\square$

Lemma 5. *The constituents of $W \otimes V_\lambda$ are the dominant $V_{\lambda+e_j}$ and $V_{\lambda-e_j}$, each once. For odd n there is also one V_λ precisely when $\lambda_\ell > 0$. There is no additional constituent for even n .*

Proof. For even $n \geq 4$, the vector representation is minuscule, so the assertion is the minuscule tensor rule. This also includes $D_2 = A_1 \times A_1$.

For odd n , multiply the Weyl alternant $A_{\lambda+\rho}$ by the vector character $1 + \sum_j (e^{e_j} + e^{-e_j})$. The result is the sum of alternants at $\lambda + \rho$ and $\lambda + \rho \pm e_j$. Adjacent coordinates of $\lambda + \rho$ differ by positive integers. A shift violating an adjacent dominant inequality lies on a wall after adding ρ , and its alternant vanishes.

At the last coordinate there are three cases. If $\lambda_\ell = 0$, the shift by $-e_\ell$ produces $-1/2$; reflection to $1/2$ has negative sign and cancels the zero-weight term. If $\lambda_\ell = 1/2$, the shift lies on the wall at zero and vanishes. If $\lambda_\ell \geq 1$, it produces an ordinary dominant term. Division by the Weyl denominator proves the formula, including $B_1 = \mathfrak{so}(3)$. In particular, the type B vector representation has not been assumed minuscule. $\square$

4 Proof of the sharp rectangular norm

By (5), the Gram eigenvalue on $V_\mu \subset W \otimes V_\lambda$ is

$$m + n - 1 + q(\lambda) - q(\mu). \quad (7)$$

For a minus shift,

$$q(\lambda) - q(\lambda - e_j) = 2\lambda_j + n - 2j - 1 \leq m + n - 3.$$

For a plus shift, the difference is $-2\lambda_j - n + 2j - 1$. It is negative if $\lambda_j \geq 0$. The only possibly negative coordinate of a dominant weight is the last coordinate in type D. For that coordinate the difference is at most $m - 1 \leq m + n - 3$. The additional type B constituent has difference zero. Thus every eigenvalue in (7) is at most $2(m + n - 2)$.

To attain the bound, consider the dominant weights

$$\lambda^{(r)} = \begin{cases} (r/2, 0, \dots, 0), & r \text{ even}, \\ (r/2, 1/2, \dots, 1/2), & r \text{ odd}. \end{cases}$$

Starting from zero, obtain this path by alternately adding $\nu_+ = (1/2, 1/2, \dots, 1/2)$ and $\nu_- = (1/2, -1/2, \dots, -1/2)$, beginning with ν_+ . Every intermediate weight is dominant. The minuscule rule therefore puts $V_{\lambda^{(m)}}$ in $\Delta_n^{\otimes m}$. For even n , both half-spin modules are available, so no sign-parity restriction removes a step.

If $m \geq 2$, then $\mu = \lambda^{(m)} - e_1$ is dominant. Lemma 5 puts it in $W \otimes V_{\lambda^{(m)}}$. Since $\lambda_1^{(m)} = m/2$, equation (7) gives exactly $2(m + n - 2)$. This proves Theorem 1 for $m \geq 2$, $n \geq 3$.

When $n = 2$, write $B = \sum_i c_{i1}c_{i2}$. Then

$$C_{m,2}^* C_{m,2} = \begin{pmatrix} m & B \\ -B & m \end{pmatrix}.$$

The operators $-ic_{i1}c_{i2}$ are commuting self-adjoint unitaries. Every joint sign projector has canonical trace $2^{-m} > 0$, so $-iB$ has largest eigenvalue m . The Gram matrix has largest eigenvalue $2m$. Finally $C_{1,n}C_{1,n}^* = nI$ and $C_{m,1}^*C_{m,1} = mI$. These give all remaining cases. Signed permutations of the generators induce Clifford star automorphisms and preserve the matrix norm. $\square$

5 The circular moment limit

For any labels $u_1, \dots, u_{2k}$ the canonical Clifford trace satisfies the signed Wick identity

$$\tau_N(c_{u_1} \cdots c_{u_{2k}}) = \sum_{\pi \in \mathcal{P}_2(2k)} (-1)^{\text{cr}(\pi)} \prod_{\{a,b\} \in \pi} \delta_{u_a, u_b}. \quad (8)$$

Here $\mathcal{P}_2$ denotes pairings and cr counts crossings. To prove the identity, commute the first generator through the word using (1). After taking the trace, the last term is the negative of the original trace by cyclicity. Dividing by two gives the Wick recursion. Induction proves (8), including repeated labels. Odd words have trace zero because the canonical trace is grading-invariant.

Expand a normalized trace of a word of length $2k$ in $X_N^{(r)}$ and their adjoints. A pairing contributes only when paired positions have the same color r . It equates the ordered row-column labels at those positions; at an adjointed position the cyclic edge is reversed. If v vertex classes remain after the identifications, the contribution is exactly

$$(-1)^{\text{cr}(\pi)} N^{v-k-1}.$$

The quotient of the cyclic walk is connected and has at most k edges, whence $v \leq k + 1$.

Equality holds precisely for noncrossing pairings with opposite adjoint status at each pair. Indeed, equality makes the quotient a tree with exactly k edges. Each edge is traversed twice by the closed walk, once in each direction. Equality of ordered generator labels then forces opposite adjoint status. A leaf gives an adjacent paired out-and-back segment; removing leaves recursively shows that the pairing is noncrossing. Conversely, successively remove adjacent pairs of a noncrossing pairing. Opposite adjoint status identifies the two outer vertices and leaves the middle vertex free, giving $v = k + 1$ by induction.

The surviving pairings are therefore exactly the free circular Wick pairings: colors match, adjoint status is opposite, and crossings are absent. Their signs are all positive. Every other term tends to zero. This proves the moment assertion in Theorem 3.

6 A row polynomial detects the norm defect

Horizontal concatenation of the d arrays is $N^{-1/2} C_{N,dN}$ up to a generator permutation. Consequently Theorem 1 gives

$$\left\| \sum_{r=1}^d X_N^{(r)} (X_N^{(r)})^* \right\| = N^{-1} \|C_{N,dN}\|^2 = 2(d+1) - \frac{4}{N}$$

for $N \geq 2$. Notice that the normalization is $1/N$, not $1/(dN)$.

In the standard full-Fock realization write $x_r = \ell_{2r-1} + \ell_{2r}^*$, with creation isometries having mutually orthogonal ranges. For the row operator $R = [x_1 \cdots x_d]$, the creation part has norm 1: $\sum_r \ell_{2r-1} \ell_{2r-1}^*$ is a nonzero projection. The annihilation part has norm $\sqrt{d}$ because $\sum_r \ell_{2r}^* \ell_{2r} = dI$. The triangle inequality gives $\|R\| \leq 1 + \sqrt{d}$, proving (4). Since

$$2(d+1) - (1 + \sqrt{d})^2 = (\sqrt{d} - 1)^2 > 0 \quad (d \geq 2),$$

strong convergence fails. This finishes Theorem 3. For two arrays, a strict finite- N separation already occurs at $N = 24$: $6 - 4/24 = 35/6 > 3 + 2\sqrt{2}$, as $(17/6)^2 - 8 = 1/36 > 0$.

7 Reproducibility and limits of verification

The supplementary source package contains two checkers. The integer and rational checker performs 84,228 checks: Clifford commutators and Gram identities, Weyl dimensions, tensor decompositions and eigenvalues for $1 \leq m \leq 8$, $3 \leq n \leq 10$, the pairing-graph exponent rule through word length eight, signed Wick traces, and negative controls. Normal and optimized Python runs give byte-identical reports. A separate NumPy checker builds faithful Jordan–Wigner matrices for 16 sizes, through 2 by 7, 3 by 5, and 4 by 4; the maximum top-eigenvalue discrepancy is below $6 \cdot 10^{-14}$. The common exact-character and numerical spectra agree. These are finite regression checks, not a substitute for the all-dimension proofs above.

This manuscript is an unrefereed preprint. Its arguments were self-audited; independent review and proof-assistant verification have not been completed. AI assistance was used in mathematical exploration, literature comparison, proof checking, computational scripts and preparation of the manuscript. The author is responsible for the claims and their stated scope.

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