Nonuniformity of Free-Gradient Heat Semigroups under Finite Fisher Information

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Abstract

Let a nonempty finite bounded selfadjoint tuple have finite joint nonmicrostates free Fisher information, and let A be the positive selfadjoint operator associated with the closure of its polynomial free-gradient form. We prove that e^{-tA} does not converge uniformly to the identity in L^2 on the operator-norm unit ball as $t \downarrow 0$. The conclusion holds for every positive number of variables, including one and two. For each coordinate, oscillatory unitaries e^{isX_i} have energy asymptotic to a positive constant times s, while $\|Ae^{isX_i}\|_2 = O(s)$. The latter bound follows from the joint conjugate-variable formula and an L^3 estimate for a one-sided Fejér operator. Resolvent duality, or independently a spectral-moment argument, yields an explicit positive defect at every positive time. We include a direct proof of the marginal L^3-density estimate and verify the exponential generator domains. The resulting deformation is admissible for Peterson's L^2-rigidity; for at least two variables, established factoriality and nonamenability results then give primeness. No compact-resolvent or nonamenability-set hypothesis is used in the semigroup theorem. No absolute priority claim is made.

1 Introduction and statement

The 2006 AIM workshop problem list *Free Analysis* asks whether the completely positive semigroup associated with $\sum_i \partial_{X_i}^* \partial_{X_i}$ can converge uniformly to the identity on the unit ball when the generating tuple has finite free Fisher information [1]. The question appears on page 3, Section 0.3. Its paragraph does not specify a norm, but the analogous derivation question on page 7 explicitly uses L^2 on the operator-norm unit ball. We use that tracial-semigroup interpretation. Pointwise strong continuity is automatic and is not the uniformity at issue.

Let (M, τ) be a finite von Neumann algebra with a faithful normal trace state, so $\tau(1) = 1$. Let

$$M = W^*(X_1, \dots, X_m), \qquad 1 \le m < \infty,$$

where the X_i are bounded and selfadjoint. On $\mathcal{P} = \mathbb{C}\langle X_1, \dots, X_m\rangle$, write

$$\partial_i X_j = \delta_{ij}(1 \otimes 1)$$

for the free difference quotients, with the outer bimodule action on

$$H = L^2(M, \tau) \,\bar{\otimes}\, L^2(M, \tau), \qquad a(b \otimes c)d = ab \otimes cd.$$

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We assume that the joint conjugate variables

$$\xi_i = \partial_i^*(1 \otimes 1) \in L^2(M, \tau), \qquad \Phi^*(X_1, \dots, X_m) = \sum_{i=1}^m \|\xi_i\|_2^2 < \infty,$$

exist in the usual nonmicrostates sense. In particular, these are derivatives on polynomials in the entire tuple, not only on each individual coordinate algebra. The standard conjugate-variable theory and the induced differential operators are discussed in [8, 3, 2].

Let δ be the closure of the common polynomial column

$$\delta_0 = (\partial_1, \dots, \partial_m) : \mathcal{P} \subset L^2(M) \longrightarrow H^{\oplus m},$$

and let

$$A = \delta^*\delta, \qquad \mathcal{E}(x) = \|\delta x\|^2, \quad x \in \mathrm{Dom}(\delta).$$

This specifies the positive selfadjoint realization, rather than treating the formal sum as an operator defined everywhere on M. Set

$$R_t = (I + tA)^{-1}, \qquad T_t = e^{-tA},$$

and define the defects

$$\gamma(t) = \sup_{\substack{x \in M \\ \|x\|_\infty \le 1}} \|(I - R_t)x\|_2, \tag{1}$$

$$\beta(t) = \sup_{\substack{x \in M \\ \|x\|_\infty \le 1}} \|(I - T_t)x\|_2. \tag{2}$$

Here $M_1 = \{x \in M : \|x\|_\infty \le 1\}$ denotes the operator-norm unit ball. Norms of scalar densities below are Lebesgue norms on $\mathbb{R}$; norms of elements of M or its correspondences are tracial Hilbert-space norms.

Theorem 1.1. *Under the preceding hypotheses, the law of every X_i has a density $\rho_i \in L^3(\mathbb{R})$. Let $K_3 < \infty$ be an operator-norm bound on $L^3(\mathbb{R})$ for the negative half-line Fourier projection, with Fourier transform $\widehat{f}(z) = \int_{\mathbb{R}} e^{-izx} f(x)\,dx$. Put*

$$c_i = 2\pi \|\rho_i\|_{L^2(\mathbb{R})}^2, \qquad C_i = \|\xi_i\|_2 + 4\pi K_3 \|\rho_i\|_{L^3(\mathbb{R})}^{3/2}. \tag{3}$$

Then $0 < c_i \le C_i < \infty$, and, for every $t > 0$,

$$\boxed{1 \ge \beta(t) \ge \gamma(t) \ge \frac{c_i}{C_i} > 0.} \tag{4}$$

More precisely, the unitaries $u_s = e^{isX_i}$ satisfy

$$\liminf_{s \to \infty} \|(I - R_t)u_s\|_2 \ge \frac{c_i}{C_i} \qquad (t > 0).$$

Consequently the specified free-gradient heat semigroup is not uniformly continuous at zero from the operator-norm unit ball into $L^2(M)$.

The theorem uses neither freeness of the coordinates nor bounded, second, or Lipschitz conjugate variables. It does not require a nonamenability set in the polynomial core, a compact resolvent, or invariance of $W^*(X_i)$ under the full semigroup.

Peterson introduced L^2-rigidity using precisely this type of derivation deformation and proved tensor-product and normalizer obstructions [6]. Dabrowski proved factoriality and nonamenability under finite joint Fisher information for at least two variables [3]. Dabrowski–Ioana's Corollary 1.2 establishes non-L^2-rigidity under bounded first and second conjugate variables for at least two generators, or under merely finite Fisher information for at least three [4]. Their latter argument uses liberation and a nonamenable algebra generated by the remaining coordinates. Here we prove nonuniformity directly for the specified free-gradient generator, with no restriction excluding two variables. The precise rigidity consequences are verified in Section 7.

2 The polynomial form and its adjoint

We record the domain facts used below. The conjugate-variable identity is

$$\tau(\xi_i p) = (\tau \otimes \tau)(\partial_i p), \qquad p \in \mathcal{P}. \tag{5}$$

The conjugate variables are selfadjoint in L^2: apply the identity to p^*, use the flip-star relation for the derivative, and use density of polynomials to identify $\xi_i^* = \xi_i$.

For $a, b \in \mathcal{P}$, the Leibniz rule and (5) give

$$\partial_i^*(a \otimes b) = a\xi_i b - (\mathrm{id} \otimes \tau)(\partial_i a)b - a(\tau \otimes \mathrm{id})(\partial_i b). \tag{6}$$

Every term belongs to $L^2(M)$, since polynomial factors are bounded. Thus polynomial tensors belong to the adjoint domain. They are dense in H, proving closability of each ∂_i. This assertion is not an assertion that polynomial tensors form an adjoint graph core.

The finite column is closable on its common domain as well. Indeed, if $p_n \to 0$ in L^2 and $(\partial_i p_n)_i \to (\eta_i)_i$, coordinatewise closability gives $\eta_i = 0$ for every i. This justifies the closure δ used in Theorem 1.1; it does not silently replace its domain by a larger intersection of individually closed domains.

The map $J(a \otimes b) = b^* \otimes a^*$, applied componentwise, is the usual real structure of the coarse correspondence. It satisfies $\delta_0(p^*) = J\delta_0(p)$. The real closable derivation construction therefore makes $\mathcal{E}$ a completely Dirichlet form. Its unital core contains 1 with $\delta_0(1) = 0$, so the associated heat semigroup is conservative. In particular T_t restricts to a normal, unital, trace-preserving completely positive map on M. These realization facts are the standard tracial derivation framework recalled in [6, Section 2] and [3, Section 1].

3 A direct marginal density estimate

Existence of a one-variable conjugate variable implies an L^3 density; see [5, Proposition 8.18], where this direction is attributed to Belinschi–Bercovici. We give the bounded-support argument and the exact estimate needed here. This avoids confusing that converse with the sufficient-condition direction of the earlier Hilbert-transform formula.

Lemma 3.1. *Let μ be a compactly supported probability measure with a conjugate variable $\eta \in L^2(\mu)$. Then*

$$d\mu(x) = \rho(x)\,dx, \qquad \rho \in L^3(\mathbb{R}), \qquad \|\rho\|_3^3 \le \frac{3}{4\pi^2}\|\eta\|^2_{L^2(\mu)}.$$

Proof. The polynomial conjugate identity has the form

$$\int \eta(x) f(x)\, d\mu(x) = \iint \frac{f(x) - f(y)}{x - y}\, d\mu(x) d\mu(y),$$

with diagonal value $f'(x)$. It makes η real-valued: its imaginary part is orthogonal to all real polynomials, which are dense in the real $L^2(\mu)$ space.

The identity extends to C^1 functions on any compact interval containing the support. To see this, approximate the derivative uniformly by polynomials, integrate, and fix the additive constant. Both the functions and their derivatives converge uniformly. Their divided differences, including the diagonal derivatives, converge uniformly on the spectral square.

For $\operatorname{Im} z > 0$, write

$$G(z) = \int \frac{d\mu(x)}{z - x}, \qquad G_\nu(z) = \int \frac{\eta(x)\, d\mu(x)}{z - x}.$$

The resolvent function has divided difference $[(z-x)(z-y)]^{-1}$. Its conjugate identity therefore says

$$G_\nu(z) = G(z)^2. \tag{7}$$

Let

$$P_\varepsilon(x) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}$$

and put

$$p_\varepsilon = P_\varepsilon * \mu = -\frac{\operatorname{Im} G(\,\cdot + i\varepsilon)}{\pi}, \qquad q_\varepsilon = \frac{\operatorname{Re} G(\,\cdot + i\varepsilon)}{\pi}.$$

Taking imaginary parts of (7) gives

$$P_\varepsilon * (\eta\mu) = 2\pi p_\varepsilon q_\varepsilon.$$

Pointwise Cauchy–Schwarz against the positive measure $P_\varepsilon(x - y)\, d\mu(y)$ yields

$$|P_\varepsilon * (\eta\mu)|^2 \le p_\varepsilon \big(P_\varepsilon * (\eta^2 \mu)\big).$$

Since $p_\varepsilon > 0$, division and integration, using the unit mass of the Poisson kernel, imply

$$4\pi^2 \int_{\mathbb{R}} p_\varepsilon q_\varepsilon^2\, dx \le \|\eta\|_{L^2(\mu)}^2. \tag{8}$$

For completeness, the integral of $G(z)^3$ on a horizontal line of positive height is zero. Close the line by a large upper semicircle. The function is analytic inside, and compact support gives $G(z)^3 = O(|z|^{-3})$ uniformly on the arc, whose integral tends to zero. The horizontal integral is absolutely convergent. Taking the imaginary part of $G^3 = \pi^3 (q_\varepsilon - i p_\varepsilon)^3$ proves

$$\int_{\mathbb{R}} p_\varepsilon^3\, dx = 3 \int_{\mathbb{R}} p_\varepsilon q_\varepsilon^2\, dx. \tag{9}$$

Equations (8) and (9) give a uniform bound on p_ε in $L^3(\mathbb{R})$.

Take a weakly convergent subsequence in this reflexive space as $\varepsilon \downarrow 0$. The measures $p_\varepsilon\, dx$ converge to μ against compactly supported continuous functions by the Poisson approximate identity. Such functions also belong to the dual space $L^{3/2}(\mathbb{R})$, so the weak L^3 limit represents μ. It is the required density ρ. Weak lower semicontinuity gives the claimed bound. $\square$

For $X = X_i$, let $B = W^*(X)$. The trace-preserving conditional expectation of ξ_i onto $L^2(B)$ is the marginal conjugate variable: restricting (5) to polynomials in X proves the identity, and conditional expectation contracts L^2. Lemma 3.1 therefore gives

$$d\mu_i = \rho_i \, dx, \qquad \|\rho_i\|_3^3 \leq \frac{3}{4\pi^2}\|\xi_i\|_2^2. \tag{10}$$

Interpolation with $\|\rho_i\|_1 = 1$ gives $\rho_i \in L^2$ and $\|\rho_i\|_2 > 0$. In particular the selected coordinate is neither scalar nor atomic. No pointwise principal-value formula alone is being substituted for a conjugate-variable identity.

4 Oscillatory unitaries and the full generator

Fix a coordinate i, abbreviate $X = X_i$, $\xi = \xi_i$, $\rho = \rho_i$, and put

$$u_s = e^{isX} \qquad (s > 0).$$

Lemma 4.1. *For every $s > 0$, $u_s \in \mathrm{Dom}(A)$. Its closed-column derivative is*

$$\partial_i u_s = i \int_0^s e^{ivX} \otimes e^{i(s-v)X} \, dv, \qquad \partial_j u_s = 0 \quad (j \neq i). \tag{11}$$

Proof. For a polynomial in X, the i-th derivative is its scalar divided difference on the spectral square. Uniform C^1 polynomial approximation of $x \mapsto e^{isx}$ thus converges in the common column graph norm and gives (11). In particular $u_s \in \mathrm{Dom}(\delta)$. The integral is a Bochner integral in H.

It remains to verify the adjoint domain, rather than stopping at the form domain. For $a = e^{ivX}$ and $b = e^{iwX}$, approximate a, b by polynomials in uniform C^1 norm on a containing interval. The tensors converge in H. Their divided differences converge uniformly on the spectral square, so the two contraction terms in (6) converge in norm. The terms involving the joint conjugate variable converge in L^2; for example,

$$\|(a_n - a)\xi b_n\|_2 \leq \|a_n - a\|_\infty \|\xi\|_2 \|b_n\|_\infty,$$

and the analogous estimate treats $a\xi(b_n - b)$. Closedness of ∂_i^* extends (6) to these exponential tensors.

On the finite interval $0 \leq v \leq s$, the tensors $e^{ivX} \otimes e^{i(s-v)X}$ and their adjoint images are continuous in their respective Hilbert norms, by the same formula and boundedness of X. They are therefore continuous in the adjoint graph norm. Integrating in that graph norm shows that the i-th component in (11) is in $\mathrm{Dom}(\partial_i^*)$. Embedded in the column with all other components zero, it belongs to $\mathrm{Dom}(\delta^*)$. Thus $\delta u_s \in \mathrm{Dom}(\delta^*)$, proving $u_s \in \mathrm{Dom}(A)$ for the specified common-column closure. □

Define the characteristic function

$$\varphi(z) = \tau(e^{izX}) = \int_{\mathbb{R}} e^{izx} \rho(x) \, dx.$$

Proposition 4.2. *For every $s > 0$, the full generator satisfies*

$$\begin{aligned} Au_s &= i \int_0^s e^{ivX} \xi e^{i(s-v)X} \, dv \\ &\quad + 2u_s \int_0^s (s - z)\varphi(z) e^{-izX} \, dz. \end{aligned} \tag{12}$$

Proof. Lemma 4.1 permits application of the adjoint under the integral in (11). The first term in (6) gives the first term in (12). Each contraction has a plus sign, since the outside i multiplies the negative i supplied by that contraction.

Explicitly, the first contraction is

$$\begin{aligned}\int_0^s \int_0^v e^{iwX} \varphi(v-w) e^{i(s-v)X} \, dw \, dv& \\ &= u_s \int_0^s \int_0^v \varphi(v-w) e^{-i(v-w)X} \, dw \, dv \\ &= u_s \int_0^s (s-z) \varphi(z) e^{-izX} \, dz.\end{aligned}$$

The second contraction is

$$\begin{aligned}\int_0^s \int_0^{s-v} e^{ivX} \varphi(w) e^{i(s-v-w)X} \, dw \, dv& \\ &= u_s \int_0^s \int_0^{s-v} \varphi(w) e^{-iwX} \, dw \, dv \\ &= u_s \int_0^s (s-w) \varphi(w) e^{-iwX} \, dw.\end{aligned}$$

The changes of order concern finite integrals of continuous bounded functions. The two contractions coincide, giving exactly the factor 2 in (12). □

The joint conjugate variable ξ need not commute with X. Nevertheless, unitary left and right multiplication give

$$\left\| \frac{i}{s} \int_0^s e^{ivX} \xi e^{i(s-v)X} \, dv \right\|_2 \leq \|\xi\|_2. \tag{13}$$

Thus the only remaining issue in estimating Au_s/s is the scalar Fourier term.

5 A one-sided Fejér estimate and energy growth

We use throughout this section

$$\widehat{f}(z) = \int_{\mathbb{R}} e^{-izx} f(x) \, dx, \qquad f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{izx} \widehat{f}(z) \, dz.$$

In particular $\varphi(z) = \widehat{\rho}(-z)$. Let P_- denote the Fourier multiplier $1_{(-\infty,0)}$. The real-line Riesz projection theorem gives a finite constant K_3 such that

$$\|P_- f\|_{L^3(\mathbb{R})} \leq K_3 \|f\|_{L^3(\mathbb{R})}. \tag{14}$$

Only boundedness is needed, not a sharp value. The Hilbert-transform multiplier and its L^p boundedness, which give (14), are proved in [7, Proposition 1.2 and Corollary 2.10]; rescaling the Fourier frequency does not change this half-line projection.

Lemma 5.1. *For*

$$F_s(x) = \int_0^s (1 - z/s) \varphi(z) e^{-izx} \, dz,$$

one has

$$\|F_s\|_{L^3(\mathbb{R})} \leq 2\pi K_3 \|\rho\|_{L^3(\mathbb{R})}, \qquad \|F_s(X)\|_2 \leq 2\pi K_3 \|\rho\|_{L^3(\mathbb{R})}^{3/2}.$$

Proof. The real-line Fejér kernel

$$k_s(x) = \frac{1-\cos(sx)}{\pi s x^2}, \qquad k_s(0) = \frac{s}{2\pi},$$

is nonnegative, has integral 1, and has Fourier transform $(1-|z|/s)_+$. These assertions follow by inverting the triangular function; its inverse is the displayed nonnegative kernel, and its value at frequency zero is its integral.

Fourier inversion and the substitution $z \mapsto -z$ give the exact identity

$$F_s = 2\pi P_-(k_s * \rho). \tag{15}$$

Indeed, the negative-frequency part of the inverse transform has integrand $(1-|z|/s)\widehat{\rho}(z)e^{izx}$ for $-s < z < 0$, which becomes the defining integral for F_s. Equality first holds in L^2. The frequency integrand is compactly supported and integrable, so that formula also gives its continuous representative. The singleton frequency zero contributes no additional mode.

Young's inequality and (14) imply

$$\|F_s\|_3 \le 2\pi K_3 \|k_s * \rho\|_3 \le 2\pi K_3 \|\rho\|_3.$$

To pass to the tracial norm, use the spectral law of X and Hölder's inequality with exponents $3/2$ and 3:

$$\|F_s(X)\|_2^2 = \int_{\mathbb{R}} |F_s(x)|^2 \rho(x)\, dx \le \|F_s\|_3^2 \|\rho\|_3.$$

This proves the second assertion with its density factor. □

Combining (12), (13), and Lemma 5.1 gives

$$\boxed{\|Au_s/s\|_2 \le \|\xi\|_2 + 4\pi K_3 \|\rho\|_3^{3/2} = C_i, \qquad s > 0.} \tag{16}$$

Lemma 5.2. *With $E_s = \|\delta u_s\|^2$, one has*

$$E_s = \int_0^s \int_0^s |\varphi(v-w)|^2\, dv\, dw = 2\int_0^s (s-z)|\varphi(z)|^2\, dz. \tag{17}$$

Consequently

$$\frac{E_s}{s} \longrightarrow 2\pi \|\rho\|_2^2 = c_i > 0, \qquad E_s \le c_i s. \tag{18}$$

Proof. The inner product of the Duhamel tensor at v with the one at w is

$$\tau(e^{i(v-w)X})\, \tau(e^{i(w-v)X}) = |\varphi(v-w)|^2.$$

Taking the squared norm in (11) and grouping by $v-w$ proves (17). Since $\rho \in L^2(\mathbb{R})$, Plancherel gives

$$2\int_0^\infty |\varphi(z)|^2\, dz = \int_{\mathbb{R}} |\varphi(z)|^2\, dz = 2\pi \|\rho\|_2^2.$$

Dominated convergence applied to $2(1-z/s)_+|\varphi(z)|^2$ proves the limit; its pointwise bound by $2|\varphi(z)|^2$ gives the stated inequality. □

As a normalization crosscheck, the scalar difference-quotient formula expresses E_s/s by the kernel

$$\frac{4\sin^2(sh/2)}{sh^2}.$$

This is an approximate identity of total mass 2π, acting on the autocorrelation of ρ, whose value at zero is $\|\rho\|_2^2$. It yields the same constant. Also $E_s/s = \langle u_s, Au_s/s\rangle \le \|Au_s/s\|_2$, since $\|u_s\|_2 = 1$, so (16) and (18) show $c_i \le C_i$.

6 Resolvent witnesses and spectral tails

Proof of Theorem 1.1. The density assertion and finiteness and positivity of the constants were proved above. Fix i and set

$$\zeta_s = \frac{\delta u_s}{s}.$$

Lemma 4.1 gives $\zeta_s \in \mathrm{Dom}(\delta^*)$, and the preceding estimates give

$$\delta^*\zeta_s = \frac{Au_s}{s}, \qquad \|\zeta_s\|^2 = \frac{E_s}{s^2} \le \frac{c_i}{s} \longrightarrow 0, \qquad \|\delta^*\zeta_s\|_2 \le C_i. \tag{19}$$

For each fixed $t > 0$, scalar functional calculus yields

$$\|\delta R_t\| = \|A^{1/2}(I+tA)^{-1}\| \le \sup_{\lambda\ge 0} \frac{\sqrt{\lambda}}{1+t\lambda} = \frac{1}{2\sqrt{t}}. \tag{20}$$

In particular $R_t u_s \in \mathrm{Dom}(\delta)$. Adjoint duality and $\|u_s\|_2 = 1$ imply

$$\begin{aligned}|\langle R_t u_s, \delta^*\zeta_s\rangle| &= |\langle \delta R_t u_s, \zeta_s\rangle| \\ &\le \frac{\sqrt{c_i}}{2\sqrt{ts}} \longrightarrow 0 \quad (s\to\infty),\end{aligned} \tag{21}$$

whereas

$$\langle u_s, \delta^*\zeta_s\rangle = \frac{E_s}{s} \longrightarrow c_i.$$

Subtracting, taking absolute values, and applying Cauchy–Schwarz with (19) therefore give

$$\liminf_{s\to\infty} \|(I-R_t)u_s\|_2 \ge \frac{c_i}{C_i}.$$

Each u_s belongs to the operator-norm unit ball, proving $\gamma(t) \ge c_i/C_i$.

For $\lambda \ge 0$,

$$\frac{t\lambda}{1+t\lambda} \le 1 - e^{-t\lambda} \le 1.$$

Squaring the scalar multipliers inside the spectral integral of the same selfadjoint operator A proves

$$\|(I-R_t)x\|_2 \le \|(I-T_t)x\|_2 \le \|x\|_2.$$

Since $\|x\|_2 \le 1$ for $x \in M_1$, this proves (4). No operator-monotonicity assertion about the exponential is involved.

The order of limits is important: for each fixed $t > 0$, first take $s\to\infty$. The resulting lower bound is independent of t; then $t \downarrow 0$ cannot give uniform convergence. No estimate uniform in unrestricted simultaneous s, t limits is needed. ☐

6.1 An independent spectral-moment argument

The last step can also be checked without the vectors ζ_s. Let ν_s be the spectral measure of A at the unit vector u_s. Lemma 4.1 and the established estimates give

$$\int \lambda \, d\nu_s(\lambda) = E_s \sim c_i s, \qquad \int \lambda^2 \, d\nu_s(\lambda) = \|Au_s\|_2^2 \le C_i^2 s^2.$$

For every fixed $L > 0$, split the first moment at L and use Cauchy–Schwarz on the high part:

$$\frac{E_s}{s} \le \frac{L}{s} + C_i \|1_{(L,\infty)}(A)u_s\|_2.$$

Thus

$$q(L) := \sup_{x \in M_1} \|1_{(L,\infty)}(A)x\|_2 \ge \frac{c_i}{C_i} \qquad (L > 0). \tag{22}$$

The scalar multiplier bounds on the high spectral subspace give

$$\beta(t) \ge (1 - e^{-tL})q(L), \qquad \gamma(t) \ge \frac{tL}{1 + tL} q(L).$$

Letting $L \to \infty$ for fixed $t > 0$ recovers the same two lower bounds. This also locates the obstruction in high spectral tails of the von Neumann unit ball.

7 Rigidity and primeness consequences

Corollary 7.1. *Under Theorem 1.1, M is not L^2-rigid in the sense of Peterson. If $m \ge 2$, then M is a nonamenable prime II_1 factor. In particular it cannot have a diffuse regular subalgebra whose inclusion has relative property* (T)*.*

Proof. We verify that the particular derivation, rather than an unspecified deformation, fits the definition. In [6, Definition 4.1], an L^2-deformation is obtained from a real closable derivation on a weakly dense star algebra in an ambient finite tracial algebra, with values in a countable direct sum of its coarse correspondence. Take that ambient algebra to be M itself. Section 2 verifies the required domain, closability, and reality. The finite column embeds in a countable coarse sum by adding zero coordinates, which changes neither the closed form nor its generator.

Peterson's resolvent parameter is

$$\eta_\alpha = \alpha(\alpha + A)^{-1} = R_{\alpha^{-1}}.$$

The topology is the same L^2 norm on M_1; [6, Lemma 2.1] also identifies heat and resolvent uniformity. The lower bound in Theorem 1.1 consequently provides an admissible nonuniform deformation and negates L^2-rigidity. The definition does not require a nonamenability set in this derivation's polynomial domain.

For $m \ge 2$, factoriality and nonamenability follow from [3, Theorems 1 and 13], in their at-least-two-variable setting. We apply the explicit nonamenable-factor statement [6, Corollary 4.6]: a nonamenable II_1 factor that is nonprime is L^2-rigid. Its contrapositive now gives primeness. Equivalently, if $M = P \bar{\otimes} Q$ with both factors diffuse, at least one is nonamenable. Peterson's Theorem 4.3 makes the deformation uniform on the other factor, and his Theorem 4.5 passes uniformity through its normalizer to all of M, a contradiction.

Finally, a relative-property-(T) inclusion forces uniformity on its unit ball for the deformations in question. If that subalgebra is diffuse and regular, the same normalizer theorem forces uniformity on M_1. This contradicts the already constructed deformation and proves the last assertion. □

Remark 7.2. *The primeness implication uses the nonamenability qualification in Peterson's Corollary 4.6, rather than relying on the broader abbreviated wording of Theorem 1.1 in that paper's introduction. Both the qualification and its verification are important. Primeness here excludes a tensor decomposition into two diffuse factors; it does not exclude finite matrix amplification. When $m = 1$, M is diffuse abelian, so the L^2-rigidity assertion still makes sense, but the factor and primeness conclusions do not apply.*

Remark 7.3. *No conclusion about solidity or absence of Cartan subalgebras is drawn from Corollary 7.1. Such conclusions require additional arguments. In particular the new semigroup proof should not be read as importing the stronger regularity hypotheses or the stronger structural conclusions of other results in [4].*

8 Scope and verification boundary

The nonempty tuple condition is essential to the question considered here. An empty tuple would generate the scalar algebra with zero generator. Under the actual hypotheses, a scalar selected coordinate is impossible already from the conjugate identity, and the density lemma gives the stronger atomlessness conclusion.

The proof uses a marginal density but never compresses the full resolvent to the marginal algebra. In general $W^*(X_i)$ need not be invariant under R_t or T_t. The term

$$i \int_0^s e^{ivX_i} \xi_i e^{i(s-v)X_i} \, dv$$

retains the *joint* conjugate variable throughout. Its estimate requires no commutation assumption. This is why the argument applies to all finite numbers of generators, rather than only to the one-variable closed form.

The witnesses are complex unitaries, which are legitimate elements of M_1. Even if uniformity were tested only on selfadjoint contractions, their real and imaginary parts give an obstruction: both are selfadjoint contractions, and the triangle inequality forces at least one of their defects to be at least half the displayed unitary defect. Translating or nontrivially scaling a coordinate changes the constants but preserves their finiteness and positivity. No tuple-independent positive lower bound is asserted.

The conclusion is not an application of the generally false implication that an unbounded derivation must have a nonuniform semigroup. Its input is the stronger paired information

$$\mathcal{E}(u_s) \sim c_i s, \qquad \|Au_s\|_2 \le C_i s$$

for bounded unitary witnesses, together with the verified operator domains. The proof is analytic and does not infer an infinite- dimensional assertion from a finite computation.

The comparison with the older three-variable finite-Fisher branch of [4] concerns the method and the stated hypotheses, not a claim of contradiction with that result. Likewise, concatenating tuples from different tensor factors does not produce a counterexample: such a tuple has cross-commutation relations and need not have finite *joint* Fisher information. Adding free semicircular noise changes the ambient or generated algebra, so primeness of a regularized algebra does not identify it with the original one.

The marginal density theorem, the real-line Riesz projection bound, and the rigidity implications are established inputs, cited separately from the oscillatory argument. This preprint makes no absolute priority claim about their combination or the resulting semigroup statement. The manuscript is unrefereed; the proof and its structural consequences remain subject to ordinary mathematical scrutiny.

AI-assisted tools supported research, source verification, independent adversarial proof checking, and manuscript preparation. No claim of formal verification or external peer review is made. The mathematical claims and final text remain the author's responsibility.

References

[1] Problems. American Institute of Mathematics, Free Analysis workshop problem list, August 2006. Dated 24 August 2006; Section 0.3, p. 3. `https://aimath.org/WWN/freeanalysis/freeanalysis.pdf`.

[2] Marwa Banna and Tobias Mai. Hölder continuity of cumulative distribution functions for noncommutative polynomials under finite free Fisher information. *Journal of Functional Analysis*, 279(8):108710, 2020.

[3] Yoann Dabrowski. A note about proving non-Γ under a finite non-microstates free Fisher information assumption. *Journal of Functional Analysis*, 258(11):3662–3674, 2010.

[4] Yoann Dabrowski and Adrian Ioana. Unbounded derivations, free dilations, and indecomposability results for II_1 factors. *Transactions of the American Mathematical Society*, 368(7):4525–4560, 2016.

[5] James A. Mingo and Roland Speicher. *Free Probability and Random Matrices*, volume 35 of *Fields Institute Monographs*. Springer, New York, 2017.

[6] Jesse Peterson. L^2-rigidity in von Neumann algebras. *Inventiones Mathematicae*, 175(2):417–433, 2009.

[7] Terence Tao. Lecture notes 4 for 247a. UCLA Math 247A: Fourier Analysis, lecture notes, 2006. Proposition 1.2, p. 3; Corollary 2.10 and following paragraph, p. 9. `https://www.math.ucla.edu/~tao/247a.1.06f/notes4.pdf`.

[8] Dan Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory. V. noncommutative Hilbert transforms. *Inventiones Mathematicae*, 132(1):189–227, 1998.