

Polynomial Rigidity and a Sharp Degree Bound for Free Quadratic Variation

Alper Ferudun*

28 September 2026

Abstract

We isolate an elementary highest-degree obstruction to polynomial changes of variables preserving unit free-Brownian diffusion. For every unital moment functional and every nonconstant noncommutative polynomial of degree d , its free quadratic-variation polynomial has degree exactly $2d - 2$. The leading coefficients form a positive Gram matrix of last-letter derivatives. At any algebraically free self-adjoint tuple, this classifies self-adjoint polynomial unit-covariance maps as affine coisometries. For a standard semicircular m -tuple, the distance of the covariance from the scalars is at least $1/\sqrt{m}$ times the squared norm of the polynomial's highest Wick component. The constant is sharp, and for quadratic polynomials this is a sharp distance-to-affine estimate. The stochastic application concerns a prescribed pathwise transformation in the same filtration; it does not resolve the more general terminal-law steering question in the AIM Free Analysis problem list.

1 The covariance polynomial

The free Itô product rule of Biane and Speicher is

$$dS_i a dS_k = \delta_{ik} \tau(a) dt, \quad (1)$$

where the S_i are freely independent standard Brownian motions and a belongs to the past algebra. See [2] and the author survey [3, Proposition 3.7 and Theorem 3.11]. We use this established rule to study a purely polynomial question.

Let $\mathcal{A} = \mathbb{C}\langle X_1, \dots, X_m \rangle$, where $m \geq 1$ and $X_i^* = X_i$. The involution conjugates coefficients and reverses words. The free difference quotient is

$$\partial_i(X_{j_1} \cdots X_{j_d}) = \sum_{\substack{1 \leq k \leq d \\ j_k = i}} X_{j_1} \cdots X_{j_{k-1}} \otimes X_{j_{k+1}} \cdots X_{j_d}.$$

We view the tensors here as vector-space tensors; their action on a noise increment is $(a \otimes b) \# dS = a dS b$.

Fix a unital complex-linear functional $\mu : \mathcal{A} \rightarrow \mathbb{C}$. For expansions

$$\partial_i P = \sum_r a_{ir} \otimes b_{ir}, \quad \partial_i Q = \sum_s c_{is} \otimes d_{is},$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.

define, by bilinearity in the tensor factors,

$$\Gamma_\mu(P, Q) = \sum_{i,r,s} a_{ir} \mu(b_{ir} d_{is}^*) c_{is}^*, \quad \Gamma_\mu(P) = \Gamma_\mu(P, P). \quad (2)$$

The tensor expansions need not be unique; the displayed sesquilinear contraction is well-defined on $\mathcal{A} \otimes \mathcal{A}$. When μ is the law of a self-adjoint tuple A with independent unit free-Brownian noises, (1) makes $\Gamma_\mu(P, Q)(A)$ the coefficient of $dP(A) dQ(A)^*$. This is an operator-valued coefficient, not its trace.

Write $[P]_d$ for the homogeneous degree- d part of a polynomial. The first result uses only $\mu(1) = 1$; positivity and traciality of μ are unnecessary for its algebraic conclusion.

Theorem 1.1 (Exact degree). *Let $P \in \mathcal{A}$ have degree $d \geq 1$, and write uniquely*

$$[P]_d = \sum_{i=1}^m R_i X_i, \quad R_i \text{ homogeneous of degree } d-1.$$

Then

$$[\Gamma_\mu(P)]_{2d-2} = \sum_{i=1}^m R_i R_i^* \neq 0, \quad \deg \Gamma_\mu(P) = 2d-2. \quad (3)$$

Consequently $\Gamma_\mu(P)$ is scalar if and only if P is affine, including constant polynomials for which $\Gamma_\mu(P) = 0$.

Proof. Every left factor in $\partial_i P$ has degree at most $d-1$. The only terms attaining that degree remove the last letter of a word in $[P]_d$; their sum is $R_i \otimes 1$. In (2), a term has outer degree $\deg a_{ir} + \deg c_{is}$. The inner moment is a scalar. Thus the degree- $2d-2$ terms come precisely from $R_i \otimes 1$ on both sides, proving the equality in (3).

Write $R_i = \sum_{|w|=d-1} c_{wi} w$. The coefficient of ww^* in $\sum_i R_i R_i^*$ is

$$\sum_{i=1}^m |c_{wi}|^2. \quad (4)$$

Indeed, a word of length $2d-2$ has a unique split into two halves of length $d-1$. Some coefficient in (4) is positive because $[P]_d \neq 0$. This also handles $d=1$, when the halves are empty words. The degree assertion follows. An affine polynomial gives the scalar sum of the squared moduli of its linear coefficients; constants give zero. $\square$

2 Unit covariance and pathwise changes

A self-adjoint tuple $A = (A_1, \dots, A_m)$ is called *algebraically free* here if evaluation $P \mapsto P(A)$ is injective on $\mathcal{A}$. This is not a claim of free independence for its joint law.

Corollary 2.1 (Polynomial classification). *Let A be algebraically free in a tracial W^* -probability space $(\mathcal{M}, \tau)$, with law μ . Suppose the self-adjoint polynomials $F_1, \dots, F_r \in \mathcal{A}$ satisfy*

$$\Gamma_\mu(F_j, F_k)(A) = \delta_{jk} 1 \quad (1 \leq j, k \leq r). \quad (5)$$

Then, as polynomial identities,

$$F_j = c_j + \sum_{i=1}^m Q_{ji} X_i, \quad c \in \mathbb{R}^r, \quad Q \in M_{r,m}(\mathbb{R}), \quad QQ^T = I_r.$$

Conversely these maps satisfy (5). In particular $r \leq m$; when $r = m$, Q is orthogonal.

Proof. Injective evaluation lifts each diagonal identity in (5) to $\Gamma_\mu(F_j) = 1$ in $\mathcal{A}$. Theorem 1.1 forces F_j to be affine and nonconstant. Self-adjointness makes its coefficients real. Substitution in (2) gives $\Gamma_\mu(F_j, F_k) = \sum_i Q_{ji} Q_{ki}$, proving both directions. $\square$

Remark 2.2. Unit diagonal covariances alone force affine coordinates with unit rows. They do not force different rows to be orthogonal: the pair $(F_1, F_2) = (X_1, X_1)$ has unit diagonal and nonzero cross covariance.

Here is a precise setting for the pathwise consequence. Work in a fixed filtered tracial W^* -probability space. Let $A_i(t)$ be norm-continuous, locally norm-bounded, adapted self-adjoint processes with locally norm-integrable drifts, satisfying

$$dA_i(t) = dS_i(t) + k_{i,t} dt. \quad (6)$$

The Brownian family is standard relative to the given filtration. These regularity assumptions are a convenient setting for the usual polynomial Itô product formula; no general existence theorem is asserted here.

Corollary 2.3 (Pathwise rigidity). *Let $F = (F_1, \dots, F_m)$ be a fixed self-adjoint polynomial map and $B(t) = F(A(t))$. Suppose, in the same filtration,*

$$dB_j(t) = dT_j(t) + \ell_{j,t} dt \quad (7)$$

with standard free Brownian family T and locally norm-integrable drift. If $A(t_0)$ is algebraically free at one interior time t_0 , then $F(X) = QX + c$ for $Q \in O(m)$ and $c \in \mathbb{R}^m$.

Conversely such maps preserve standard diffusion, allowing a rotation of the Brownian family. If the original drift is a functional of the history, the transformed history drift is explicitly

$$\ell_t(B_{\leq t}) = Q k_t(Q^\top(B_{\leq t} - c)). \quad (8)$$

Proof. The martingale term in the polynomial product formula is $\sum_i (\partial_i F_j)(A(t)) \# dS_i(t)$. Finite-variation terms have zero quadratic covariation. Comparing (6)–(7) by (1) gives

$$\Gamma_{\mu_t}(F_j, F_k)(A(t)) = \delta_{jk} 1$$

for almost every t . Every expression on the left is norm-continuous: it is a finite sum of polynomial expressions in $A(t)$ and their scalar moments. Hence the identity holds at t_0 too, and Corollary 2.1 applies. Without norm continuity one must instead assume an algebraically free time in the full-measure set where the covariance identity holds.

For the converse, take $T = QS$. Orthogonal combinations of a semicircular increment family are again standard, and its increments remain free from the past. Substituting $A = Q^\top(B - c)$ gives (8). $\square$

If the same coupling requires literally $T = S$, the free Itô isometry forces $Q = I_m$. Rotation of a standard noise family is not equality with the old family. For rectangular coisometries, a unit-diffusion representation still exists in the original filtration, but the drift need not be expressible from the shorter output history alone.

The freeness assumption is automatic for a standard semicircular tuple at every positive Brownian time. To see this, use full Fock space and $S_i = l_i + l_i^*$, where l_i creates the letter i . The degree- d component of $P(S)\Omega$ consists of the degree- d word coefficients of P applied to orthonormal creation words. A nonzero highest component therefore cannot evaluate to zero. Positive variances only rescale it.

3 A sharp quantitative obstruction

In this section $S = (S_1, \dots, S_m)$ is a freely independent standard semicircular tuple of variance one, and μ is its law. Let H_k denote orthogonal projection in L^2 onto degree- k free Wick chaos. Wick words W_w have leading monomial X_w and satisfy $W_w(S)\Omega = e_w$; in particular they are orthonormal. These facts follow directly from

$$W_{iw} = X_i W_w - \delta_{i,w_1} W_{\text{tail}(w)}, \quad W_\emptyset = 1,$$

with the subtraction omitted when w is empty.

Theorem 3.1 (Sharp highest-degree bound). *Let P have degree $d \geq 2$. Put*

$$E_d(P) = \sum_{|v|=d} |[v]P|^2 = \|H_d(P(S))\|_2^2.$$

For every scalar $a \in \mathbb{C}$,

$$\|\Gamma_\mu(P)(S) - a1\|_2 \geq \frac{E_d(P)}{\sqrt{m}}. \quad (9)$$

The constant $1/\sqrt{m}$ cannot be increased uniformly over $d \geq 2$.

Proof. Use the coefficients c_{wi} in the proof of Theorem 1.1, and form the matrix $C = (c_{wi})_{|w|=d-1, 1 \leq i \leq m}$. Its Gram matrix $G = CC^*$ is positive semidefinite, has rank at most m , and satisfies $\text{Tr } G = E_d(P)$. The top coefficients of $\Gamma_\mu(P)$ are G_{wv} , indexed by the distinct words wv^* . Lower-degree terms do not contribute to degree- $2d-2$ Wick chaos. Thus

$$\|H_{2d-2}(\Gamma_\mu(P)(S))\|_2 = \|G\|_{\text{HS}} \geq \frac{\text{Tr } G}{\sqrt{m}}.$$

The last inequality is Cauchy–Schwarz on the at most m nonzero eigenvalues of G . Since $2d-2 > 0$, scalar subtraction leaves this projection unchanged, proving (9).

For sharpness take $P = \sum_{i=1}^m X_i^2$. Directly from (2),

$$\Gamma_\mu(P)(S) = \sum_{i=1}^m S_i^2 + m1.$$

For $a = 2m$ the remainder is $\sum_i (S_i^2 - 1)$, of norm $\sqrt{m}$, whereas $E_2(P) = m$. Equality holds. $\square$

Corollary 3.2 (Quadratic distance to affine maps). *If $\deg P \leq 2$, and $V = \text{span}_{\mathbb{C}}\{1, S_1, \dots, S_m\}$, then*

$$\text{dist}_2(P(S), V) \leq m^{1/4} \text{dist}_2(\Gamma_\mu(P)(S), \mathbb{C}1)^{1/2}. \quad (10)$$

The constant is sharp.

Proof. For degree two the distance on the left is exactly $\|H_2(P(S))\|_2$; apply Theorem 3.1 and minimize over a . For degree at most one the left side is zero. The same quadratic example gives equality. $\square$

For larger degree, (9) controls only the highest component. We do not deduce a degree-independent bound for the entire nonlinear part.

4 Boundary cases, provenance, and scope

The hypotheses distinguish several apparently similar claims.

Traced covariance is too weak. For $m = 2$ and $P = (X_1^2 + X_2^2)/2$,

$$\Gamma_\mu(P)(S) = \frac{S_1^2 + S_2^2 + 2}{4}, \quad \tau(\Gamma_\mu(P)(S)) = 1, \quad \|\Gamma_\mu(P)(S) - 1\|_2^2 = \frac{1}{8}.$$

Thus normalized scalar energy is not the operator identity required above.

Algebraic relations permit nonlinear representatives. At the one-time tuple $A = 0$, the nonlinear polynomial $P = X + X^3$ has $\partial P(A) = 1 \otimes 1$. For a two-point spectrum with $A^2 = 1$, $P = X + (X^2 - 1)^2$ has $P(A) = A$ and again $\partial P(A) = 1 \otimes 1$. Its divided difference is one at all pairs in $\{-1, 1\}^2$, including the diagonal derivatives. These are failures of injective evaluation, not examples of nonlinear changes preserving Brownian diffusion throughout an interval.

Complex coefficients do not evade the obstruction. The polynomial $i(X_1X_2 - X_2X_1)$ is self-adjoint. Its leading covariance is $X_1^2 + X_2^2$, in agreement with the positive diagonal certificate (4).

Relation to the AIM question. Section 0.2, p. 2, of the AIM Free Analysis list [1] asks for a drift transformation realizing a transformed *terminal law*. The frozen dataset entry AIM-PROBABILITY-0108 also contains two adjacent questions about entropy limits and pressure equilibria. Our result deals only with polynomial *pathwise* changes $B = F(A)$ in the same filtration, and does not rule out another process with the desired terminal law. Neither adjacent question is resolved here.

The inherited dataset report supplied a one-variable interval-support obstruction and the affine orthogonal converse. The elementary degree argument extends the polynomial classification under algebraic freeness and gives the quantitative estimate. All foundational stochastic and Fock-space tools are credited to the existing literature. An adjacent affine-rigidity theorem of Diez [4, Theorem 3.15] assumes saturation of a free Poincaré inequality, Lipschitz conjugate variables, and a curvature-dimension bound. Those hypotheses are different from the operator-valued covariance identity here; its splitting conclusions are not asserted under our hypotheses.

Verification and disclosure. The source package includes an exact Gaussian-rational checker. It compares Fock creation–annihilation moments with noncrossing-pair recursion, and raw Itô contractions with the Gram formula. It checks 240 polynomials in up to four variables and degree at most five, including 120 self-adjoint samples, as well as the sharp and boundary examples. Its 4003 explicit checks pass with two Python runtimes and with optimization enabled. These finite checks supplement the written all-degree proof; they are not formal verification or independent review.

The author used generative-AI assistance in developing and checking the arguments and preparing the text, and remains responsible for the manuscript. This elementary note is unrefereed. A bounded literature search did not identify the precise degree and quantitative statements; the possibility that the rigidity observation is folklore is not excluded, and no absolute-priority claim is made.

References

- [1] American Institute of Mathematics, *Free Analysis: Problems*, 24 August 2006, section 0.2, p. 2.
<https://aimath.org/WWN/freeanalysis/freeanalysis.pdf>.
- [2] P. Biane and R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probab. Theory Related Fields* **112** (1998), 373–409.
[doi:10.1007/s004400050194](https://doi.org/10.1007/s004400050194).
- [3] R. Speicher, *Free Calculus*, 2001. <https://arxiv.org/abs/math/0104004>.
- [4] C.-P. Diez, *Obata's rigidity theorem in free probability*, preprint, 2026.
<https://arxiv.org/abs/2603.05466>.