

Hidden Constants after Free Semicircular Regularization

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Abstract

A bounded conjugate variable does not imply a coarse relative free Poincaré inequality. Let B be a diffuse tracial von Neumann algebra, let Y be a bounded self-adjoint element commuting with B , and put $A = W^*(B, Y)$. If S is a variance-one semicircular element free from A , then $X = Y + \varepsilon S$, $\varepsilon > 0$, generates $A * W^*(S)$ together with B . The variable X is algebraically free from B and has bounded conjugate variable S/ε , but $\ker \bar{\partial}_{X:B} = L^2(A)$. Thus the kernel is strictly larger than $L^2(B)$ whenever $Y \notin B$. For a tensor Bernoulli choice of Y , equal-trace pinching polynomials have squared Poincaré ratios $n + \varepsilon^2$. This gives a negative answer to Question 7.4 of Ito's arXiv version 3 and shows that the hypotheses of the AIM X -constants question alone do not force its kernel identity. It does not classify all cases where that identity holds. We also construct an irreducible inclusion of II_1 factors with the same obstruction: for a Bernoulli crossed product $A = B \rtimes \mathbb{Z}$, the variable $X = u + u^* + \varepsilon S$ has $B' \cap W^*(B, X) = \mathbb{C}$ and closed derivative kernel $L^2(A)$. Twisted pinching polynomials have exact squared ratios $2^m + 1 + \varepsilon^2$.

1 The question and the construction

Let (M, τ) be a finite von Neumann algebra with faithful normal tracial state and $B \subset M$ a unital von Neumann subalgebra. If $X = X^* \in M$ is algebraically free from B , the free difference quotient

$$\partial_{X:B} : B\langle X \rangle \longrightarrow L^2(M, \tau) \bar{\otimes} L^2(M, \tau)$$

is the derivation determined by $\partial_{X:B} X = 1 \otimes 1$ and $\partial_{X:B} b = 0$ for $b \in B$. The target is the *ordinary coarse correspondence*, with $a \cdot (u \otimes v) \cdot b = au \otimes vb$ and $\|u \otimes v\|_2 = \|u\|_2 \|v\|_2$. It is not a tensor product balanced over B .

The AIM problem [1, Section 0.1] assumes infinite-dimensional B , $M = W^*(B, X)$ and closability of this derivation. It asks for conditions ensuring that its closed kernel is $L^2(B)$, and for conditions yielding

$$\|P - E_B P\|_2 \leq C \|\partial_{X:B} P\|_2 \quad (P \in B\langle X \rangle). \quad (1)$$

Here E_B is the trace-preserving conditional expectation, also viewed as the orthogonal projection on L^2 .

Ito [2, Section 7] gives a commuting obstruction without a conjugate variable, then asks in Question 7.4 whether a conjugate variable suffices for (1). The numbering in this note refers specifically to arXiv:2409.16498v3. We add free semicircular noise to a commuting variable and show explicitly that the hidden constant remains recoverable.

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Theorem 1. *Let (A, τ) be a finite tracial von Neumann algebra, let $B \subset A$ be a diffuse unital von Neumann subalgebra, and let $Y = Y^* \in B' \cap A$ satisfy $A = W^*(B, Y)$. Form the tracial free product*

$$(N, \tau) = (A, \tau) * (W^*(S), \tau),$$

where S is centered semicircular of variance one. For $\varepsilon > 0$, put $X = Y + \varepsilon S$. Then:

1. $W^*(B, X) = N$, and X is algebraically free from B .
2. $\partial_{X:B}$ is closable and its conjugate variable is $J(X : B) = S/\varepsilon \in N$.
3. Writing $D = \bar{\partial}_{X:B}$, one has

$$\ker D = L^2(A), \quad \|\xi - E_A \xi\|_2 \leq \varepsilon \|D\xi\|_2 \quad (\xi \in \text{dom } D). \quad (2)$$

The constant ε is optimal.

4. $S, J(X : B) \in \text{dom } D$ and

$$DS = \varepsilon^{-1} 1 \otimes 1, \quad DJ(X : B) = \varepsilon^{-2} 1 \otimes 1.$$

If $Y \notin B$, the kernel is strictly larger than $L^2(B)$ and (1) fails for every finite C .

Diffuseness here is a hypothesis on the coefficient algebra, not on the distribution of Y . In the most concrete example Y has a two-point distribution.

Corollary 2. *Take*

$$B = L^\infty([0, 1]), \quad A = B \bar{\otimes} \mathbb{C}^2, \quad Y = 1 \otimes (1, -1),$$

with Lebesgue trace on B and uniform trace on $\mathbb{C}^2$. For an equal-trace partition $1 = \sum_{i=1}^n p_i$ in B , define $q_n = \sum_{i=1}^n p_i X p_i$. Then

$$E_B q_n = 0, \quad \|q_n - Y\|_2^2 = \frac{\varepsilon^2}{n}, \quad (3)$$

$$\|q_n\|_2^2 = 1 + \frac{\varepsilon^2}{n}, \quad \|\partial_{X:B} q_n\|_2^2 = \frac{1}{n}. \quad (4)$$

In particular the squared ratio in (1) is $n + \varepsilon^2$, whereas $J(X : B)$ is bounded with $\|J(X : B)\| = 2/\varepsilon$ and $\|J(X : B)\|_2^2 = \varepsilon^{-2}$.

2 Algebraic freeness and recovery

We first record the algebraic point that distinguishes the regularized example from a commuting variable itself.

Lemma 3. *If S is semicircular and scalar-free from A , then evaluation at S embeds the algebraic free product $A * \mathbb{C}[t]$ into N . For any $Y = Y^* \in A$ and any $\varepsilon > 0$, evaluation at $Y + \varepsilon S$ is also injective.*

Proof. Centered reduced words in the two factors identify with algebraic tensors in the corresponding reduced free-product Hilbert space. The trace on A is faithful, and the semicircle law has infinite support, so no nonzero polynomial in S has zero L^2 norm. The reduced-word decomposition therefore has no algebraic kernel. The formal substitution fixing A and sending t to $Y + \varepsilon t$ is an automorphism of $A * \mathbb{C}[t]$, with inverse $t \mapsto (t - Y)/\varepsilon$. Compose it with the first injective evaluation. $\square$

In particular X is algebraically free even from A , and hence from B . This is not a claim of probabilistic freeness of X from either algebra.

Lemma 4. *In Theorem 1, let $1 = \sum_{i=1}^n p_i$ be an orthogonal partition in B with $\tau(p_i) = 1/n$, and define*

$$h_n = \sum_{i=1}^n p_i S p_i, \quad q_n = \sum_{i=1}^n p_i X p_i.$$

Then

$$E_A h_n = 0, \quad \|h_n\|_2^2 = \frac{1}{n}, \quad q_n = Y + \varepsilon h_n, \quad \|q_n\| \leq \|X\|. \quad (5)$$

Consequently $W^*(B, X) = N$.

Proof. A diffuse finite von Neumann algebra contains a diffuse abelian subalgebra and hence such equal-trace partitions. Since Y commutes with every p_i , its pinching is Y . Centering and freeness of S give $E_A S = 0$ and

$$\tau(Sa Sb) = \tau(a)\tau(b) \quad (a, b \in A).$$

The mixed terms in the square of h_n vanish by orthogonality of the projections. Thus

$$\|h_n\|_2^2 = \sum_i \tau(p_i S p_i S p_i) = \sum_i \tau(S p_i S p_i) = \sum_i \tau(p_i)^2 = \frac{1}{n}.$$

Also $E_A h_n = 0$ by bimodularity of E_A . Pinching by an orthogonal partition is a unital completely positive map, which proves the operator-norm bound.

Let $M = W^*(B, X) \subseteq N$. Since $q_n \in M$ and $q_n \rightarrow Y$ in $L^2(N)$, the L^2 -continuous conditional expectation E_M fixes Y . As Y is bounded, $Y \in M$. It follows that $S = (X - Y)/\varepsilon \in M$, so M contains A and S and is all of N . $\square$

Recovery is essential: it places S/ε in the algebra generated by B and X , not merely in an ambient free-product extension.

3 The closed derivative and its exact kernel

Let $\delta = \partial_{S:A}$ on $A\langle S \rangle$. Semicircular integration by parts gives

$$\tau(SP) = (\tau \otimes \tau)(\delta P) \quad (P \in A\langle S \rangle). \quad (6)$$

This is the standard conjugate-variable identity [5, 2]. The Leibniz rule and (6) give, on polynomial tensors,

$$\delta^*(a \otimes b) = a S b - (\text{id} \otimes \tau)(\delta a) b - a(\tau \otimes \text{id})(\delta b). \quad (7)$$

In particular the adjoint has dense domain, so δ is closable. Denote its closure by T . The relative semicircular Poincaré estimate [2, Corollary 6.11 and Remark 6.12], with coefficient algebra A and covariance $a \mapsto \tau(a)1$, states that

$$\|\xi - E_A \xi\|_2 \leq \|T\xi\|_2, \quad \ker T = L^2(A). \quad (8)$$

Here that covariance norm is exactly the ordinary coarse norm.

On $B\langle X \rangle \subset A\langle S \rangle$ one has $\delta X = \varepsilon 1 \otimes 1$ and $\delta b = 0$. Lemma 3 and the defining Leibniz rule therefore imply

$$\partial_{X:B} = \varepsilon^{-1} \delta|_{B\langle X \rangle}, \quad D \subseteq \varepsilon^{-1} T. \quad (9)$$

The domain on the left is dense in $L^2(N)$ by Lemma 4. The restriction is closable, so the notation D is justified. Equations (8)–(9) show that $\ker D \subseteq L^2(A)$.

For the reverse inclusion, direct differentiation gives

$$\partial_{X:B} q_n = \sum_i p_i \otimes p_i, \quad \|\partial_{X:B} q_n\|_2^2 = \sum_{i,j} \tau(p_i p_j)^2 = \frac{1}{n}. \quad (10)$$

Together with $q_n \rightarrow Y$ this proves $Y \in \text{dom } D$ and $DY = 0$. We spell out why the whole algebra A becomes constant. For each fixed $k \geq 1$, the uniform operator-norm bound in (5) and the telescoping identity give $q_n^k \rightarrow Y^k$ in L^2 . The coarse bimodule estimate and the Leibniz rule give

$$\|\partial_{X:B}(q_n^k)\|_2 \leq k \|X\|^{k-1} \|\partial_{X:B} q_n\|_2 \rightarrow 0. \quad (11)$$

For $b \in B$, the same reasoning applies to bq_n^k . Thus $bY^k \in \ker D$. Because Y commutes with B , the span of these elements, including $k = 0$, is $B[Y]$. It is L^2 dense in $A = W^*(B, Y)$. A closed operator has a closed kernel; hence $L^2(A) \subseteq \ker D$. This proves the kernel identity.

The inequality in (2) now follows from (8) and (9). Further, $S = (X - Y)/\varepsilon \in \text{dom } D$ and $DS = \varepsilon^{-1} 1 \otimes 1$. Since $\|S\|_2 = 1$ and $E_A S = 0$, equality for S proves that the constant ε is optimal. The argument only uses the restriction in (9); it does not assume equality of the full operator domains.

Finally, for $P \in B\langle X \rangle$, (6) gives

$$\tau((S/\varepsilon)P) = (\tau \otimes \tau)(\partial_{X:B} P).$$

This identifies $S/\varepsilon \in N = M$ as the conjugate variable. Its derivative is $\varepsilon^{-2} 1 \otimes 1$ by the preceding computation. If $Y \notin B$, then $\|q_n - E_B q_n\|_2 \rightarrow \|Y - E_B Y\|_2 > 0$, while (10) tends to zero. This proves failure of (1) and completes Theorem 1.

4 Exact ratios and comparison with earlier results

In the tensor Bernoulli example, $E_B Y = 0$ and $\|Y\|_2 = 1$. Equation (5) implies $E_B h_n = 0$ and $\langle Y, h_n \rangle = 0$. Hence

$$\|q_n\|_2^2 = \|Y + \varepsilon h_n\|_2^2 = 1 + \frac{\varepsilon^2}{n}.$$

The recovery and derivative formulas prove Corollary 2. In particular no unspecified constants or asymptotic error terms enter the obstruction.

More generally, for any finite orthogonal partition with $t_i = \tau(p_i)$, the same computations replace $1/n$ by $\sum_i t_i^2$. The distinction between $\sum_i t_i = 1$ and $\sum_i t_i^2 \rightarrow 0$ is the mechanism of the construction. The commuting component survives pinching, while both the free noise and the coarse derivative disappear in L^2 .

Ito's commuting example [2, Theorem 7.2] establishes the obstruction before regularization; his Proposition 7.3 explains why it lacks a conjugate variable. The free noise here removes that defect, and the recovery sequence shows why it does not restore the proposed inequality. This conclusion concerns the coarse norm and does not contradict the stronger projective-norm estimate of [3].

There is also a general structural antecedent. Lee and Nelson [4, Proposition 2] show that the anticoarse space associated with the constants of a closable coarse derivation is annihilated by its closure. That absorption result explains why a diffuse commuting extension can contribute extra constants. We do not claim that principle as new. Our argument gives an explicit regularized realization, its bounded conjugate variable, its exact closed kernel and a polynomial sequence with exact diverging ratios.

The result answers the conjugate-variable sufficiency question negatively. The original AIM formulation asks for conditions; a characterization of all positive cases is not provided. For example, if $Y \in B$, then $A = B$ and (2) is precisely a positive case.

5 An irreducible inclusion of factors

The preceding example has a large relative commutant. We now show that irreducibility does not restore the proposed inequality.

Theorem 5. *Let*

$$B = \overline{\bigotimes_{j \in \mathbb{Z}} (M_2(\mathbb{C}), \text{tr}_2)}, \quad A = B \rtimes_{\alpha} \mathbb{Z},$$

where α shifts tensor coordinate j to $j + 1$, and write u for the canonical unitary satisfying $ubu^* = \alpha(b)$. In $N = A * W^*(S)$, with S centered variance-one semicircular, put $X = u + u^* + \varepsilon S$ for $\varepsilon > 0$. Then

$$W^*(B, X) = N, \quad B' \cap N = \mathbb{C}.$$

The variable X is algebraically free from B , its conjugate variable is S/ε , and its closed coarse derivative satisfies

$$\ker D = L^2(A) \supsetneq L^2(B), \quad \|\xi - E_A \xi\|_2 \leq \varepsilon \|D\xi\|_2 \quad (\xi \in \text{dom } D).$$

The constant is optimal and $DJ(X : B) = \varepsilon^{-2} 1 \otimes 1$. In particular $B \subset N$ is an irreducible inclusion of II_1 factors with no finite Poincaré constant relative to B .

Here and below all tensor factors of B have normalized trace. For $m \geq 1$, let p_w , $w \in \{0, 1\}^m$, be the diagonal cylinder projections on coordinates $1, \dots, m$, and put $n = 2^m$. They partition identity and have trace $1/n$. Define

$$Q_m = \sum_w \alpha(p_w) X p_w, \quad H_m = \sum_w \alpha(p_w) S p_w, \quad F_m = \sum_w \alpha(p_w) \alpha^{-1}(p_w). \quad (12)$$

The summands of F_m are disjoint diagonal projections. On the binary coordinates $0, \dots, m + 1$, their union is the event that the strings on $0, \dots, m - 1$ and $2, \dots, m + 1$ coincide. Its m constraints force two-periodicity; there are exactly four assignments out of 2^{m+2} . Therefore

$$F_m^2 = F_m = F_m^*, \quad \tau(F_m) = 2^{-m} = 1/n. \quad (13)$$

Using $u p_w = \alpha(p_w) u$ and $u^* p_w = \alpha^{-1}(p_w) u^*$ gives

$$Q_m = u + F_m u^* + \varepsilon H_m. \quad (14)$$

In particular the unwanted u^* term is small in L^2 , not zero.

Since $E_A H_m = 0$, it is orthogonal to the first two terms. Orthogonality of the left projections and the scalar-free second-moment identity give

$$\|H_m\|_2^2 = \sum_w \tau(p_w S \alpha(p_w) S p_w) = \sum_w \tau(p_w) \tau(\alpha(p_w)) = 1/n.$$

Also u and $F_m u^*$ have distinct Fourier degrees in the crossed product. Thus

$$E_B Q_m = 0, \quad \|Q_m - u\|_2^2 = \frac{1 + \varepsilon^2}{n}, \quad (15)$$

$$\|Q_m\|_2^2 = 1 + \frac{1 + \varepsilon^2}{n}, \quad \|\partial_{X:B} Q_m\|_2^2 = \frac{1}{n}. \quad (16)$$

For the last identity differentiate directly: $\partial_{X:B} Q_m = \sum_w \alpha(p_w) \otimes p_w$. Both partitions are orthogonal, so the coarse squared norm is $\sum_w \tau(\alpha(p_w)) \tau(p_w) = 1/n$.

We first justify that this derivative is well-defined and that the bounded conjugate variable belongs to the generated algebra. Lemma 3 applies to $Y = u + u^*$, without any commuting assumption. For $M = W^*(B, X)$, (15) and L^2 continuity of E_M give $u \in M$. Thus $A \subseteq M$ and $S = (X - u - u^*)/\varepsilon \in M$, proving $M = N$. As before $D \subseteq \varepsilon^{-1}T$, where $T = \bar{\partial}_{S:A}$, and hence $\ker D \subseteq L^2(A)$.

The map $Z \mapsto \sum_w \alpha(p_w) Z p_w$ is operator-norm contractive, although it need not be positive: on any vector η , orthogonal ranges give

$$\left\| \sum_w \alpha(p_w) Z p_w \eta \right\|^2 = \sum_w \|\alpha(p_w) Z p_w \eta\|^2 \leq \|Z\|^2 \sum_w \|p_w \eta\|^2.$$

Consequently $\|Q_m\| \leq \|X\|$. Equations (15)–(16) put u in $\ker D$. The real-derivation identity $\partial(P^*) = \partial(P)^\dagger$, where $(a \otimes b)^\dagger = b^* \otimes a^*$, also puts u^* there. For any fixed word in B, u, u^* , replace u, u^* by Q_m, Q_m^* . The uniform norm bound gives L^2 convergence by telescoping, and the Leibniz rule makes the derivatives tend to zero. These words are L^2 dense in A , so $\ker D = L^2(A)$. The argument proving the sharp inequality and the identities for S and J in Section 3 now applies verbatim with $Y = u + u^*$. The exact squared ratio in (1) is $n + 1 + \varepsilon^2 = 2^m + 1 + \varepsilon^2$.

It remains to verify irreducibility. Each nonzero power α^k is outer. Indeed, the trace-zero unitary $v_j = \text{diag}(1, -1)$ in coordinate j asymptotically commutes in L^2 with every fixed element of B , by approximation with finite tensors. If $\alpha^k = \text{Ad}(a)$ for a unitary $a \in B$, this would imply $\|\alpha^k(v_j) - v_j\|_2 \rightarrow 0$ as $j \rightarrow \infty$. But that squared norm is exactly 2, because the two tensor coordinates are distinct.

If $z \in B' \cap A$, its Fourier coefficients $a_k = E_B(z u^{-k})$ satisfy $ba_k = a_k \alpha^k(b)$ for every $b \in B$. If $k \neq 0$ and $a_k \neq 0$, both $a_k^* a_k$ and $a_k a_k^*$ are the same positive scalar multiple of identity, since B is a factor. Thus a scalar multiple of a_k is a unitary implementing α^k up to inversion, a contradiction. Also a_0 is scalar. Hence $B' \cap A = \mathbb{C}$.

For completeness, the free product does not enlarge this relative commutant. The reduced-word decomposition identifies

$$L^2(N) \ominus L^2(A) \cong L^2(A) \bar{\otimes} K \bar{\otimes} L^2(A)$$

as an A – A correspondence, where K consists of reduced words beginning and ending in centered letters from $W^*(S)$. Take a Haar unitary $v \in B$. Its powers v^r tend weakly to zero. Matrix coefficients of conjugation by v^r on elementary tensors factor as

$$\langle v^r a, a' \rangle \langle k, k' \rangle \langle b v^{-r}, b' \rangle \rightarrow 0.$$

Density and the unitary norm bound extend this to all vectors. A B -central vector in this correspondence must therefore be zero. For $z \in B' \cap N$, the vector $z - E_A z$ is B -central, so $z = E_A z \in B' \cap A = \mathbb{C}$. This proves Theorem 5.

The standard outer-action and free-product facts in this argument are not new. Lee–Nelson kernel absorption also accounts for normalizing directions; our additional conclusion is the explicit irreducible regularized construction and its exact recovery, kernel and ratios. This section strengthens the same negative sufficiency result, not the unresolved classification of positive cases.

6 Reproducibility

The accompanying standard-library Python program checks finite noncrossing semicircular moment identities, the derivative chain rule, conjugate-variable identities and pinching formulas using rational arithmetic. It passes 2,190 predicates in each of normal and optimized execution, with 864 coefficient-word tests and 27 partition/noise models. These checks supplement the written infinite-dimensional argument; they are not formal verification. An additional exact checker performs 421 predicates for the irreducible construction, comparing exhaustive binary-cylinder enumeration with constraint-graph counts for $1 \leq m \leq 12$ and testing the norms and ratios at 60 rational noise models. Both checkers run identically with and without Python optimizations.

AI tools assisted with literature searches, calculations and drafting. The author is responsible for the statements and proofs. This is an unrefereed preprint; no independent review or absolute priority certification is asserted.

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