

Uniform Height Bounds for Brownian Excursions with an Exact Area Constraint

Alper Ferudun*

October 5, 2026

Abstract

Let a standard Brownian excursion have duration T , exact area ρT , and maximum M_T , where $\rho > 0$ is fixed. We prove $\mathbb{P}(M_T > H \mid A_T = \rho T) \leq C_\rho T^{3/2} e^{-c_\rho H}$ for large T and $H \geq 1$, and consequently $\mathbb{E}[M_T^r \mid A_T = \rho T] = O((\log T)^r)$ for every $r > 0$. The proof compares increasing path functionals under exact-area conditioning and an exponential area tilt. A positivity-preserving Gaussian bridge shift gives the comparison; classical Airy-area asymptotics and elementary kernel estimates complete the bound. In particular, an excursion of duration $2N$ and exact area N converges uniformly to zero after division by $N^{1/3}$. Thus the literal normalization in an AIM question cannot produce a nondegenerate random limit on an $N^{2/3}$ time window. No sharp maximum scale, constant-scale process limit, or absolute priority claim is made.

1 The exact-area bound and its scope

Let $\mathbb{P}_T$ denote the standard Brownian excursion law of duration T , with Brownian variance parameter one. For its coordinate process X , put

$$A_T = \int_0^T X(t) dt, \quad M_T = \max_{0 \leq t \leq T} X(t).$$

Conditioning on a positive exact area always means the weakly continuous regular conditional distribution constructed by Zambotti [2, Corollary 4.3], rescaled from duration one. In particular, there is no choice of an exceptional null-event version.

Theorem 1.1. *For each $\rho > 0$ there are $C_\rho, c_\rho > 0$ and $T_\rho \geq 3$ such that*

$$\mathbb{P}_T(M_T > H \mid A_T = \rho T) \leq C_\rho T^{3/2} e^{-c_\rho H} \quad (T \geq T_\rho, H \geq 1). \quad (1)$$

For every fixed $r > 0$,

$$\mathbb{E}_T[M_T^r \mid A_T = \rho T] = O_{\rho,r}((\log T)^r). \quad (2)$$

The official AIM KPZ workshop list, item 2.18, describes a single excursion on $[-N, N]$ with area N and asks about an Airy-like fluctuation limit at the $1/3, 2/3$ scales near the edge [1]. The item is headed “Gibbs line ensembles,” but its stated model contains only this one excursion. The dataset extraction labels the item 22.18; the archived source displays 2.18.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint. CC BY 4.0.

Corollary 1.2. *Let X_N have excursion duration $2N$ and exact area N . Then, for each $r > 0$,*

$$\mathbb{E} \left[\left(N^{-1/3} \sup_{-N \leq t \leq N} X_N(t) \right)^r \right] \rightarrow 0.$$

In particular every restriction to an $N^{2/3}$ time window, with any deterministic or random center and clipped to the excursion interval, converges uniformly to zero in probability after division by $N^{1/3}$.

Subtracting a nonrandom function of the rescaled time coordinate cannot turn the vanishing random term into a nondegenerate random limit. This is the precise negative conclusion for the literal normalization. It says nothing about an additional random centering, a moving wall, several curves, or a changed area exponent such as $N^{4/3}$.

The constant-scale Ferrari–Spohn one-point picture at fixed A_T/T is already described by Agranov et al. [4, Eq. (10), Appendix C]. The small-area scale diagnosis is not presented here as a new observation. Our purpose is to give an explicit one-sided comparison and a uniform maximum estimate under an *exact* area constraint. We do not assume process-level equivalence of ensembles. Recent local equivalence theorems for conservative reference processes and one-sided or fixed-width constraints [5] are related, but are not invoked for this zero-endpoint excursion bridge on a growing interval. The logarithmic bound below is not asserted to be sharp, and no absolute novelty claim is made.

2 The matched area tilt

For $\lambda > 0$ let

$$\mathbb{Q}_{T,\lambda}(\mathrm{d}X) = Z_T(\lambda)^{-1} e^{-\lambda A_T} \mathbb{P}_T(\mathrm{d}X).$$

Write $\alpha_j > 0$ for the ordered absolute values of the negative zeros of Ai , and set $\alpha = \alpha_1$ and $E_j = 2^{-1/3} \alpha_j \lambda^{2/3}$, $E = E_1$. The classical transform [3, Eq. (80)] and Brownian scaling give

$$Z_T(\lambda) = \sqrt{2\pi} \lambda T^{3/2} \sum_{j \geq 1} e^{-E_j T}. \quad (3)$$

If f is the density of the unit-excursion area, then

$$f(x) \sim \frac{8}{81} \alpha^{9/2} x^{-5} \exp\left(-\frac{2\alpha^3}{27x^2}\right), \quad x \downarrow 0, \quad (4)$$

by [3, Eq. (93)]. In particular the prefactor is x^{-5} .

Lemma 2.1. *Choose*

$$\lambda = \lambda_\rho = \frac{4\alpha^3}{27\rho^3}. \quad (5)$$

If q_T is the area density under $\mathbb{Q}_{T,\lambda}$, then

$$q_T(\rho T) \sim \frac{2\alpha^{3/2}}{3\sqrt{2\pi}\rho^2} T^{-1/2}. \quad (6)$$

Proof. Brownian scaling gives

$$q_T(a) = \frac{e^{-\lambda a}}{Z_T(\lambda)} T^{-3/2} f(a/T^{3/2}).$$

At $a = \rho T$, equations (3)–(4) apply. The choice (5) gives $E'(\lambda) = \rho$ and

$$E - \lambda \rho = \frac{2\alpha^3}{27\rho^2}.$$

Thus the exponential factors cancel. The remaining power is $T^{-1/2}$, and its coefficient is $(8/81)\alpha^{9/2}\rho^{-5}/(\sqrt{2\pi}\lambda)$, which simplifies to the coefficient in (6). $\square$

3 An increasing-functional comparison

The path order in this section is pointwise order on $C([0, T])$. For a bounded, continuous, nonnegative increasing functional Φ , let q_T^Φ be the density of the finite measure $\mathbb{E}_{\mathbb{Q}_{T,\lambda}}[\Phi(X); A_T \in da]$. Zambotti's continuity theorem and the continuous area density show that q_T^Φ is continuous on $(0, \infty)$.

Lemma 3.1. *For every $a > 0$, $u \geq 0$ and $\lambda > 0$,*

$$q_T^\Phi(a + u) \geq e^{-\lambda u - 12au/T^3 - 6u^2/T^3} q_T^\Phi(a). \quad (7)$$

Proof. Start with a Brownian bridge from ε to ε on $[0, T]$. Its area is Gaussian with mean εT and variance $T^3/12$. Conditional Gaussian regression couples its laws at areas a and $a + u$ by the addition of

$$h_u(t) = \frac{6ut(T-t)}{T^3} \geq 0, \quad \int_0^T h_u(t) dt = u.$$

Consequently

$$\mathbb{E}[\Phi(B)\mathbf{1}_{\{B>0\}} \mid A_T = a + u] \geq \mathbb{E}[\Phi(B)\mathbf{1}_{\{B>0\}} \mid A_T = a].$$

Indeed, addition of h_u preserves positivity and cannot decrease Φ . The Gaussian area-density ratio is

$$\exp\left(-\frac{12(a - \varepsilon T)u + 6u^2}{T^3}\right).$$

Multiplying the densities by $e^{-\lambda a}$ and normalizing by the same positive-bridge tilted partition function proves the finite-bridge version of (7), with a replaced by $a - \varepsilon T$ in its Gaussian term.

To pass to excursions, first integrate this inequality against a nonnegative continuous function compactly supported in $a \in (0, \infty)$. Positive ε -to- ε bridges converge weakly to $\mathbb{P}_T$; this classical limit is also recalled in the proof of [2, Theorem 4.1]. Their exponentially tilted laws converge to $\mathbb{Q}_{T,\lambda}$, since the tilt is continuous and bounded by one on nonnegative paths, with positive limiting expectation. The integrated inequality therefore passes to the limit. It gives (7) almost everywhere in a . Both sides are continuous on $(0, \infty)$, so it holds at every positive a , for each $u \geq 0$. $\square$

Corollary 3.2. *With the matched λ in (5), for all sufficiently large T ,*

$$\mathbb{E}_T[\Phi \mid A_T = \rho T] \leq C_\rho \sqrt{T} \mathbb{E}_{\mathbb{Q}_{T,\lambda}}[\Phi]. \quad (8)$$

The same inequality holds for $\Phi = \mathbf{1}_{\{M_T > H\}}$.

Proof. At $a = \rho T$, the exponential in (7) is bounded below, uniformly for $u \in [0, 1]$ and $T \geq 1$, by $e^{-\lambda-12\rho-6}$. Thus

$$c_\rho q_T^\Phi(\rho T) \leq \int_0^1 q_T^\Phi(\rho T + u) du \leq \mathbb{E}_{\mathbb{Q}_{T,\lambda}}[\Phi].$$

Divide by $q_T(\rho T) \geq c_\rho T^{-1/2}$ from Lemma 2.1. An area tilt leaves the conditional path law at a fixed area unchanged. Finally the functions $\min\{1, n(M_T - H)_+\}$ are bounded, continuous and increasing, and increase to $\mathbf{1}_{\{M_T > H\}}$. $\square$

This result is only a comparison for increasing functionals. It is not a total-variation estimate or a comparison for arbitrary path events.

4 Maximum tails under the exponential tilt

Let $K_t(x, y)$ be the Dirichlet Feynman–Kac kernel on $(0, \infty)$ for

$$\mathcal{H}_\lambda = -\frac{1}{2} \frac{d^2}{dx^2} + \lambda x, \quad b_t(x) = \lim_{\varepsilon \downarrow 0} \frac{K_t(\varepsilon, x)}{\varepsilon}.$$

Its eigenvalues are the E_j above. For $\kappa = (2\lambda)^{1/3}$ its normalized eigenfunctions can be chosen as

$$\phi_j(x) = \frac{\sqrt{\kappa} \operatorname{Ai}(\kappa x - \alpha_j)}{|\operatorname{Ai}'(-\alpha_j)|}.$$

The identity $\int_{-\alpha_j}^\infty \operatorname{Ai}(z)^2 dz = \operatorname{Ai}'(-\alpha_j)^2$ follows by integrating the derivative of $z \operatorname{Ai}(z)^2 - \operatorname{Ai}'(z)^2$. Hence $|\phi_j'(0)|^2 = \kappa^3 = 2\lambda$, and

$$D_T := \lim_{\varepsilon, \eta \downarrow 0} \frac{K_T(\varepsilon, \eta)}{\varepsilon \eta} = 2\lambda \sum_{j \geq 1} e^{-E_j T} \geq 2\lambda e^{-ET}. \quad (9)$$

Lemma 4.1. *For fixed $\lambda > 0$, there is C_λ such that*

$$b_t(x) \leq C_\lambda e^{-Et} e^{-\lambda x/2} \quad (t \geq 1, x > 0). \quad (10)$$

Consequently, for $1 \leq s \leq T - 1$,

$$\mathbb{Q}_{T,\lambda}(X(s) > H) \leq C_\lambda e^{-\lambda H}. \quad (11)$$

Proof. The free killed entrance density bounds $b_{1/2}$, so $b_{1/2} \in L^2$. By the semigroup property, its L^2 norm bound, and Cauchy–Schwarz,

$$b_t(x) \leq \|b_{t-1/2}\|_2 \sqrt{K_1(x, x)} \leq C_\lambda e^{-Et} \sqrt{K_1(x, x)}.$$

Dropping positivity bounds $K_1(x, x)$ by the full-line bridge expectation with the same linear potential. The area of that bridge is Gaussian with mean x and variance $1/12$, giving

$$K_1(x, x) \leq (2\pi)^{-1/2} e^{-\lambda x + \lambda^2/24}.$$

This proves (10). The one-time density of the tilted excursion is $b_s(x)b_{T-s}(x)/D_T$. Apply (9) and integrate. $\square$

Lemma 4.2. *For fixed $\lambda > 0$, there are $C, c > 0$ such that*

$$\mathbb{Q}_{T,\lambda}(M_T > H) \leq C(T+1)e^{-cH} \quad (T \geq 3, H \geq 1). \quad (12)$$

Proof. Consider an interval $[i, i+1]$ with $i \geq 1$ and $T-i-1 \geq 1$. If its maximum exceeds H , either an endpoint exceeds $H/2$, an event controlled by (11), or both endpoints x, y are at most $H/2$ and the intervening bridge crosses H . On positive paths its unit-interval tilt is at most one. Dropping that tilt and positivity, reflection bounds the restricted transition density by $p_1(2H-x-y)$, where $p_1(z) = (2\pi)^{-1/2}e^{-z^2/2}$. The entrance factors and (9) therefore bound this second alternative by

$$C_\lambda \int_0^{H/2} \int_0^{H/2} e^{-\lambda(x+y)/2} p_1(2H-x-y) dx dy \leq C_\lambda e^{-H^2/2}.$$

Thus each such unit interval has maximum tail at most Ce^{-cH} .

We handle a boundary interval $[0, b]$, $1 \leq b \leq 2$, separately. The finite-endpoint version of the preceding kernel estimate is

$$K_t(x, \varepsilon) \leq C_\lambda \varepsilon e^{-Et} e^{-\lambda x/2}, \quad t \geq 1, \quad 0 < \varepsilon \leq \varepsilon_0. \quad (13)$$

Indeed, split off half a unit of time at both ends, apply the L^2 semigroup norm, and use

$$K_1(\varepsilon, \varepsilon) \leq p_1(0) - p_1(2\varepsilon) \leq C\varepsilon^2.$$

The ground-state term also gives $K_T(\varepsilon, \varepsilon) \geq c_\lambda \varepsilon^2 e^{-ET}$ for small ε . Path decomposition at b , with $T-b \geq 1$, now bounds the probability of crossing H on $[0, b]$ under the tilted positive ε -bridge by

$$\frac{C_\lambda}{\varepsilon} \mathbb{P}_\varepsilon(\tau_H \leq b, \tau_H < \tau_0).$$

Here the factor e^{Eb} is bounded uniformly for $1 \leq b \leq 2$; we have dropped the nonnegative-area tilt on this initial segment and relaxed positivity after the first hit. The symbols τ_H, τ_0 denote Brownian hitting times.

For $s > 0$ and $k = \sqrt{2s}$, the elementary exit-time equation gives

$$\mathbb{E}_\varepsilon[e^{-s\tau_H}; \tau_H < \tau_0] = \frac{\sinh(k\varepsilon)}{\sinh(kH)}.$$

It follows that

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathbb{P}_\varepsilon(\tau_H \leq b, \tau_H < \tau_0) \leq \frac{e^{sb}k}{\sinh(kH)}.$$

Take $s = H^2/(2b^2)$. The last expression is at most $CH e^{-H^2/(2b)}$, uniformly for $H \geq 1$ and $b \in [1, 2]$. Weak convergence of the tilted positive bridges and the Portmanteau theorem for the open maximum event give the same bound for the excursion. Time reversal treats the other endpoint.

For $m = \lfloor T \rfloor$, use $[0, 1]$, the intervals $[i, i+1]$ for $1 \leq i \leq m-2$, and $[m-1, T]$. The last length lies in $[1, 2]$, and every interior interval has both outside legs at least one. There are at most $T+1$ intervals. The union bound proves the claim. $\square$

Proof of Theorem 1.1 and Corollary 1.2. Combine Corollary 3.2 with Lemma 4.2 at the matched λ_ρ . This gives (1). Integrating $rH^{r-1} \min\{1, C_\rho T^{3/2} e^{-c_\rho H}\}$ over $H \geq 1$, and bounding the integral over $[0, 1]$ by one, proves (2). Finally use $T = 2N$, $\rho = 1/2$ and $(\log N)/N^{1/3} \rightarrow 0$. The same supremum bounds every time window, including windows chosen from the path. $\square$

5 Scaling and limitations

There is a useful dimensionless version. Brownian scaling identifies an excursion of duration T and exact area ρT with ρ times an excursion of duration $S = T/\rho^2$ and exact area S , after the time change $t = \rho^2 s$. Thus the unit-density constants yield

$$\mathbb{P}_T(M_T > \rho H \mid A_T = \rho T) \leq C S^{3/2} e^{-cH}, \quad S \geq S_0, H \geq 1.$$

This remains valid when ρ varies, and gives $M_T = O_{\mathbb{P}}(\rho \log S)$ as $S \rightarrow \infty$. It supplies an obstruction to vertical scales growing faster than $\rho \log S$, not a nondegenerate process limit or a sharp extreme-value asymptotic. In particular a different area exponent is a different question, not an additional conclusion of this note.

AI assistance was used in developing the argument, checking algebra and preparing this manuscript. The accompanying symbolic and numerical checks are supplementary: they are neither a formal verification of the probabilistic proof nor a certification of novelty. This preprint has not undergone independent peer review.

References

- [1] American Institute of Mathematics, *The Kardar–Parisi–Zhang equation and universality class*, Open problems, item 2.18, “Gibbs line ensembles.” <http://aimpl.org/kpzuniversality/2/>. Official page checked through its 16 October 2014 archived snapshot.
- [2] L. Zambotti, *On the Brownian meander and excursion conditioned to have a fixed time average*, arXiv:math/0605720v2 (2007). <https://arxiv.org/abs/math/0605720>.
- [3] S. Janson, Brownian excursion area, Wright’s constants in graph enumeration, and other Brownian areas, *Probability Surveys* **4** (2007), 80–145. doi:10.1214/07-PS104.
- [4] T. Agranov, P. Zilber, N. R. Smith, T. Admon, Y. Roichman and B. Meerson, Airy distribution: Experiment, large deviations, and additional statistics, *Physical Review Research* **2** (2020), 013174. doi:10.1103/PhysRevResearch.2.013174.
- [5] T. Schmidt, *Conditioned Brownian motion and local equivalence of path ensembles*, arXiv:2608.19744v1 (2026). <https://arxiv.org/abs/2608.19744>.