

Spatial Mollification of the Multilayer Stochastic Heat Equation

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Abstract

We give a spatial mollifier construction of the O’Connell–Warren multilayer stochastic heat equation directly from nonintersecting Brownian bridges with coalesced endpoints. Both the fully Wick-ordered exponential of the total path energy and the product of the separately Wick-ordered path weights converge to the chaos partition function in every finite L^p norm. The difference between the two approximations is $O(\varepsilon^\alpha)$ for every $\alpha < 1$. The proof combines the established chaos and intersection-local-time estimates with a mollifier-independent inverse-gap bound for the mutual-path counterterm. All parameters, including the number of layers, are fixed in the convergence statement.

1 The spatial smoothing problem

Let $p_t(z) = (2\pi t)^{-1/2} e^{-z^2/(2t)}$. Fix an integer $n \geq 1$, $t > 0$, $x, y \in \mathbb{R}$, and $\beta \in \mathbb{R}$. Let $Q = Q_{n,t}^{x,y}$ be the law of the Dyson bridge $X = (X_1, \dots, X_n)$ whose coordinates all start at x and all end at y . Coordinates are ordered decreasingly. This entrance-law bridge is well defined and has strict gaps for $0 < s < t$ [5, Section 2]; it is not an ordinary conditioning on a positive-probability event. Let W be white noise, independent of X , viewed as an isonormal Gaussian process over $\mathcal{H} = L^2((0, t) \times \mathbb{R})$.

Choose a nonnegative Schwartz function ρ with integral one. Thus smooth compactly supported and Gaussian mollifiers are included. Set $\check{\rho}(z) = \rho(-z)$ and

$$\rho_\varepsilon(z) = \varepsilon^{-1} \rho(z/\varepsilon), \quad R = \rho * \check{\rho}, \quad R_\varepsilon = \rho_\varepsilon * \check{\rho}_\varepsilon.$$

The smoothed occupation field and its mutual-path overlap are

$$H_{\varepsilon,X}(s, z) = \sum_{i=1}^n \rho_\varepsilon(X_i(s) - z), \quad S_\varepsilon(X) = \sum_{i < j} \int_0^t R_\varepsilon(X_i(s) - X_j(s)) ds. \quad (1)$$

There are two natural finite-scale partition functions:

$$F_\varepsilon = \mathbb{E}_Q \exp\{\beta W(H_{\varepsilon,X}) - \tfrac{1}{2} \beta^2 \|H_{\varepsilon,X}\|_{\mathcal{H}}^2\}, \quad (2)$$

$$G_\varepsilon = \mathbb{E}_Q \exp\{\beta W(H_{\varepsilon,X}) - \tfrac{1}{2} \beta^2 n t R_\varepsilon(0)\}. \quad (3)$$

The first Wick-orders the total path energy. The second multiplies the n individually Wick-ordered path weights. Both are well-defined random variables with moments of every order at each fixed $\varepsilon > 0$, since $\|H_{\varepsilon,X}\|_{\mathcal{H}}^2 \leq n^2 t R_\varepsilon(0)$.

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Let R_k be the ordered k -time occupation correlation density of Q , and let $\bar{R}_k$ be its symmetric extension to the full time cube. In the convention $\mathbb{E}I_k(f)^2 = k!\|f\|_2^2$, the O’Connell–Warren partition function is

$$Z_n^\beta(t, x, y) = p_t(x - y)^n \sum_{k \geq 0} \frac{\beta^k}{k!} I_k(\bar{R}_k). \quad (4)$$

Its L^2 convergence is established in [5, Theorem 1.2]; the general β formulation also appears in [3, Proposition 1.2].

Theorem 1. *For the parameters and mollifiers above and every $1 \leq p < \infty$,*

$$p_t(x - y)^n F_\varepsilon \longrightarrow Z_n^\beta(t, x, y), \quad p_t(x - y)^n G_\varepsilon \longrightarrow Z_n^\beta(t, x, y) \quad \text{in } L^p(W).$$

Moreover, for every $0 < \alpha < 1$,

$$\|G_\varepsilon - F_\varepsilon\|_{L^p(W)} = O(\varepsilon^\alpha).$$

Constants may depend on all the fixed parameters and on α .

The rate compares the two approximations, not their distance to the limiting chaos series. The theorem gives a spatial smoothing construction for the question in AIM’s KPZ problem list, item 2.04 [1]. The imported UnsolvedMath record AIM-PROBABILITY-0086 labels that item 22.04; the public source’s embedded number is 2.04. Finite collections of parameters converge jointly under the same white noise. Uniform sample-path convergence, temporal smoothing and limits in the layer number are not asserted.

The one-layer antecedent is the Feynman–Kac construction of Bertini–Cancrini [2]. Corwin–Nica [3, Remark 1.3] distinguish that construction from the then-untreated multilayer spatial smoothing problem; their principal approximation is by discrete nonintersecting polymers. Lun–Warren [4, Eq. (1.5)] write the multilayer bridge Wick formula as shorthand for the chaos expansion. Our argument relies substantially on O’Connell–Warren’s estimates, including the inverse-gap integrability implicit in their proof of Proposition 4.2. It supplies a direct construction at coalesced endpoints and an explicit comparison of the two Wick conventions, not a new definition of the limiting object.

2 An inverse-gap dominator

For a Dyson Brownian motion U in the closed chamber $\Lambda_n = \{u_1 \geq \cdots \geq u_n\}$, define

$$A_T(U) = \sum_{i < j} \int_0^T \frac{ds}{U_i(s) - U_j(s)}.$$

The sum is zero for $n = 1$.

Lemma 2. *For every $c, T > 0$, $\sup_{u \in \Lambda_n} \mathbb{E}_u e^{cA_T(U)} < \infty$. For the coalesced bridge X , $\mathbb{E}_Q e^{cA_t(X)} < \infty$ for every $c > 0$.*

Proof. The Dyson SDE, including boundary starts, is

$$dU_i = dB_i + \sum_{j \neq i} (U_i - U_j)^{-1} ds,$$

where the B_i are independent standard Brownian motions [5, Section 2]. Take $a_i = n + 1 - 2i$. Pairing the drift terms yields

$$2 \sum_{i < j} (j - i) \int_0^T \frac{ds}{U_i(s) - U_j(s)} = \sum_i a_i (U_i(T) - u_i) - \sum_i a_i B_i(T). \quad (5)$$

The left side is nonnegative and at least $2A_T(U)$. Represent U as the ordered eigenvalues of a Hermitian Brownian motion H with deterministic initial eigenvalues u . Weyl's inequality gives $|U_i(T) - u_i| \leq \|H_T - H_0\|_{\text{op}}$. Thus

$$A_T(U) \leq \frac{1}{2} \left(\sum_i |a_i| \|H_T - H_0\|_{\text{op}} + \left| \sum_i a_i B_i(T) \right| \right).$$

Both terms have exponential moments of all orders, uniformly in u : one is bounded by a finite-dimensional Gaussian norm, and the other is the absolute value of a Gaussian. They need not be independent; Cauchy–Schwarz suffices.

For the bridge, split the integral at $t/2$ and use Cauchy–Schwarz. On the first half its density relative to Dyson Brownian motion from $x\mathbf{1}$ is

$$\frac{q_n(t/2, U_{t/2}, y\mathbf{1})}{q_n(t, x\mathbf{1}, y\mathbf{1})}.$$

The denominator is positive and the numerator bounded [5, Lemma 2.1 and Section 2]. Apply the Brownian-motion bound; the second half is identical after time reversal. $\square$

The elementary covariance calculation

$$\|H_{\varepsilon, X}\|_{\mathcal{H}}^2 = ntR_{\varepsilon}(0) + 2S_{\varepsilon}(X) \quad (6)$$

shows why the two conventions differ at finite scale. Set $K_{\rho} = \sup_{r>0} rR(r) < \infty$. For every positive r ,

$$0 \leq R_{\varepsilon}(r) \leq K_{\rho}/r, \quad R_{\varepsilon}(r) \longrightarrow 0.$$

Strict nonintersection in the open interval and Lemma 2 therefore imply

$$S_{\varepsilon} \longrightarrow 0 \quad Q\text{-almost surely}, \quad 0 \leq S_{\varepsilon} \leq K_{\rho}A_t(X). \quad (7)$$

Dominated convergence, first in time and then in Q , gives

$$\|e^{cS_{\varepsilon}} - 1\|_{L^r(Q)} \longrightarrow 0 \quad (c > 0, 1 \leq r < \infty). \quad (8)$$

This uses a single random dominator with all exponential moments. It does not infer exponential integrability from a first-moment estimate.

3 Chaos contraction and replica moments

The published estimates needed below are

$$M(q) := \sum_{k \geq 0} \frac{q^k}{k!} \|\bar{R}_k\|_2^2 = \mathbb{E}_{X,Y} e^{qL(X,Y)} < \infty \quad (q \geq 0). \quad (9)$$

Here X, Y are independent bridges and L is their total intersection local time, in the occupation normalization of [5, Lemma 4.1]. All exponential moments follow from [5, Proposition 4.2 and the bridge reduction in Section 4]. The same proof is valid at coalesced endpoints.

Define the spatial convolution contraction on $\mathcal{H}$ by

$$(T_\varepsilon f)(s, z) = \int_{\mathbb{R}} \rho_\varepsilon(u - z) f(s, u) du.$$

It converges strongly to the identity. The correlation definition and the fixed-scale Wick expansion give

$$\mathbb{E}_Q H_{\varepsilon, X}^{\otimes k} = T_\varepsilon^{\otimes k} \bar{R}_k, \quad F_\varepsilon = \sum_{k \geq 0} \frac{\beta^k}{k!} I_k(T_\varepsilon^{\otimes k} \bar{R}_k). \quad (10)$$

Indeed, the deterministic bound on $\|H_{\varepsilon, X}\|_{\mathcal{H}}$ justifies Bochner averaging the conditional Wick series, and product test functions identify its kernels. Time diagonals are null. If $\hat{Z}^\beta = Z_n^\beta / p_t(x - y)^n$, orthogonality yields

$$\mathbb{E}_W |F_\varepsilon - \hat{Z}^\beta|^2 = \sum_{k \geq 1} \frac{\beta^{2k}}{k!} \|T_\varepsilon^{\otimes k} \bar{R}_k - \bar{R}_k\|_2^2 \longrightarrow 0.$$

The dominating summable sequence is four times that in (9) with $q = \beta^2$. The normalization agrees with ordered stochastic integration: $\|\bar{R}_k\|_2^2 / k! = \|R_k\|_2^2$.

Put $C_\varepsilon(X, Y) = \langle H_{\varepsilon, X}, H_{\varepsilon, Y} \rangle_{\mathcal{H}} \geq 0$. Tonelli, the tensor identity in (10), and contraction give

$$\mathbb{E}_{X, Y} e^{q C_\varepsilon(X, Y)} = \sum_{k \geq 0} \frac{q^k}{k!} \|T_\varepsilon^{\otimes k} \bar{R}_k\|_2^2 \leq M(q). \quad (11)$$

For an integer $m \geq 2$ let $N = m(m - 1)/2$. Conditional Gaussian integration over W , followed by Hölder's inequality over the N pairs of replicas, implies

$$\mathbb{E}_W F_\varepsilon^m = \mathbb{E}_{Q^{\otimes m}} e^{\beta^2 \sum_{a < b} C_\varepsilon(X^a, X^b)} \leq M(N\beta^2). \quad (12)$$

No independence of overlapping pairs is assumed in Hölder's inequality. Uniform arbitrarily high moments, the L^2 convergence above, Fatou's lemma and interpolation give convergence of F_ε in every finite $L^p(W)$.

4 The separately ordered weights

Let $V_\varepsilon(X)$ denote the exponential in (2) and put $D_\varepsilon(X) = e^{\beta^2 S_\varepsilon(X)} - 1$. By (6),

$$G_\varepsilon - F_\varepsilon = \mathbb{E}_Q [V_\varepsilon(X) D_\varepsilon(X)] \geq 0.$$

For an integer $m \geq 2$, positivity and Gaussian integration give the exact replica identity

$$\mathbb{E}_W (G_\varepsilon - F_\varepsilon)^m = \mathbb{E}_{Q^{\otimes m}} \left[e^{\beta^2 \sum_{i < j} C_\varepsilon(X^i, X^j)} \prod_{i=1}^m D_\varepsilon(X^i) \right]. \quad (13)$$

For conjugate exponents $a, b > 1$, apply Hölder, then (11) to the replica pairs. Independence of the replicas factors the single-path terms and yields

$$\|G_\varepsilon - F_\varepsilon\|_{L^m(W)} \leq M(aN\beta^2)^{1/(am)} \|D_\varepsilon\|_{L^b(Q)} \longrightarrow 0, \quad N = m(m - 1)/2. \quad (14)$$

The last limit is (8). Taking $m \geq p$ proves the second convergence in Theorem 1. This argument starts with the coalesced bridge itself and does not interchange a Vandermonde normalization limit with removal of the mollifier.

5 Quantitative comparison

At time $s \in (0, t)$ the bridge has a translated GUE eigenvalue density at scale $\sigma_s = \sqrt{s(t-s)/t}$. For each $i < j$, the positive scaled gap $(X_i(s) - X_j(s))/\sigma_s$ has a density satisfying

$$h_{ij}(r) \leq C_{n,ij} r^2, \quad r \geq 0.$$

To see this, retain the relevant squared Vandermonde factor, integrate the other polynomial factors against the Gaussian density, and enlarge the ordered integration region. After factoring out r^2 , the bound is a polynomial times a decaying Gaussian in r , hence is bounded. This works for adjacent and nonadjacent gaps alike. Therefore

$$\mathbb{E}_Q R_\varepsilon(X_i(s) - X_j(s)) \leq \min \left\{ \frac{\|R\|_\infty}{\varepsilon}, C \frac{\varepsilon^2}{\sigma_s^3} \int_0^\infty u^2 R(u) du \right\}.$$

For small ε , split at times within ε^2 of either endpoint. Those pieces cost $O(\varepsilon)$, and the remaining integral is bounded by a constant times $\varepsilon^2 \int_{\varepsilon^2}^{t/2} s^{-3/2} ds$. Thus

$$\mathbb{E}_Q S_\varepsilon = O(\varepsilon). \tag{15}$$

For $b > 1$ and $0 < \theta < 1$, elementary exponential bounds imply $D_\varepsilon^b \leq C S_\varepsilon^\theta e^{c S_\varepsilon}$. Using Hölder, (7) and Lemma 2,

$$\mathbb{E}_Q D_\varepsilon^b \leq C (\mathbb{E}_Q S_\varepsilon)^\theta (\mathbb{E}_Q e^{c S_\varepsilon / (1-\theta)})^{1-\theta} = O(\varepsilon^\theta).$$

Given $0 < \alpha < 1$, take $1 < b < 1/\alpha$ and $\theta = \alpha b$. Equation (14) proves the claimed $O(\varepsilon^\alpha)$ comparison in every finite $L^p(W)$. This completes the proof of Theorem 1.

For $n = 1$ or $\beta = 0$ the difference is identically zero. As a normalization check, for $n = 2$ and a standard Gaussian mollifier one can integrate the GUE gap density explicitly to obtain

$$\mathbb{E}_Q S_\varepsilon = \frac{2\varepsilon t}{\sqrt{\pi} (t + 4\varepsilon^2)}.$$

In particular the endpoint contribution cannot simply be discarded.

Scope and provenance. The proof gives the fixed-parameter construction posed in [1]. The chaos summability, Dyson entrance law and exponential-moment machinery are credited to [5]; this is a direct consequence and application of those methods. A bounded literature search found no explicit matching coalesced-endpoint spatial mollifier theorem, but does not establish absolute priority. The accompanying exact algebra checks support normalization and signs; they do not formally verify the stochastic convergence argument.

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