

Hitting Moving Targets on Eulerian Digraphs in $O(mn)$ Expected Time

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Abstract

Let G be a connected loopless Eulerian directed multigraph with n vertices and m arcs, and let X be its half-lazy simple random walk. For every deterministic target sequence $(u_t)_{t \geq 0}$, we prove $\mathbb{E}_x \inf\{t \geq 0 : X_t = u_t\} \leq t_{\text{unif}}(1/4) + 10m(n-1) = O(mn)$, uniformly in the starting vertex. This removes the logarithmic factor in the general moving-target bound of Boczkowski, Peres and Sousi and answers their explicit question following Corollary 2.3. The proof combines a point Dirichlet inequality with a contraction estimate for killed nonreversible operators and a burn-in of the unconditioned law. Conditioning on an independent target trajectory gives the same bound for independent random targets. We also give a simple nonreversible example with $n = 18$, $m = 51$ and a fixed-target expected hitting time of $936 > mn$. This example rules out coefficient one for arbitrary deterministic targets, but does not settle that coefficient for two independent walks.

1 The moving-target bound

On an Eulerian digraph, how long can a target evade a random walk if its trajectory is specified in advance? Boczkowski, Peres and Sousi [1, Corollary 2.3 and Section 2.1] proved an upper bound of $O(mn(1 + \log(m/n)))$ for a half-lazy simple walk on a general Eulerian digraph, and $O(n^2)$ on regular Eulerian digraphs. They then asked whether the general bound could be improved to $O(mn)$. The following theorem supplies that improvement.

Let $G = (V, A)$ be a finite loopless directed multigraph whose underlying undirected graph is connected. Write a_{xy} for the number of arcs from x to y , and assume the Eulerian condition

$$d(x) := \sum_y a_{xy} = \sum_y a_{yx}.$$

Set $n = |V| \geq 2$ and $m = |A| = \sum_x d(x)$, with arcs counted with multiplicity. Such a digraph is strongly connected. Its simple walk and half-lazy simple walk have respective transition matrices

$$R(x, y) = \frac{a_{xy}}{d(x)}, \quad P = \frac{I + R}{2}.$$

The probabilistic holding in P is not counted among the m arcs. Both matrices preserve the probability measure $\pi(x) = d(x)/m$. For this P , define the relative pointwise mixing time

$$T = t_{\text{unif}}(1/4) := \min \left\{ t \geq 0 : \max_{x, y \in V} \left| \frac{P^t(x, y)}{\pi(y)} - 1 \right| \leq \frac{1}{4} \right\}. \quad (1)$$

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Theorem 1.2 of [1] gives $T \leq c_1 mn$ for a universal constant c_1 . This is the only general mixing estimate used below.

Theorem 1.1. *For every deterministic sequence $u = (u_t)_{t \geq 0}$ of vertices and every starting vertex x , let*

$$\tau_u = \inf\{t \geq 0 : X_t = u_t\}, \quad X_0 = x.$$

Then

$$\mathbb{E}_x \tau_u \leq T + 10m(n-1) \leq (c_1 + 10)mn. \quad (2)$$

Moreover, for every integer $k \geq 0$,

$$\mathbb{P}_x(\tau_u > T + k) \leq \frac{5}{4} \exp\left(-\frac{k}{8m(n-1)}\right). \quad (3)$$

No adjacency condition is imposed on consecutive target vertices.

The theorem concerns arbitrary Eulerian digraphs, not just regular or vertex-transitive ones. It does not improve the separate $O(n^2)$ regular-graph bound of [1]. Its hypotheses specify half-laziness: arbitrary holding probabilities approaching one are not covered with the same constants.

Problem 3.1 in the AIM collection [3], attributed to Perla Sousi, also asks about meeting a moving target on a nonreversible Eulerian walk. Its displayed inequality has coefficient one and does not specify the same walk and target conventions. We distinguish that formulation from the explicit logarithm-removal question in [1]; Section 4 explains the distinction.

2 A contraction for killed steps

Work in the real Hilbert space $L^2(\pi)$, with $\langle f, g \rangle_\pi = \sum_x \pi(x) f(x) g(x)$. Matrices act on column functions, so $(Pf)(x) = \sum_y P(x, y) f(y)$. For $a \in V$, let D_a be the orthogonal projection

$$(D_a f)(x) = \mathbf{1}_{\{x \neq a\}} f(x).$$

Although P need not be self-adjoint, its quadratic energy $\mathcal{E}_P(f, f) = \langle f, (I - P)f \rangle_\pi$ satisfies

$$\mathcal{E}_P(f, f) = \frac{1}{2} \sum_{x, y} \pi(x) P(x, y) (f(x) - f(y))^2 = \frac{1}{4m} \sum_{x, y} a_{xy} (f(x) - f(y))^2. \quad (4)$$

Indeed, stochasticity and stationarity make the two square terms in the first sum contribute $\|f\|_\pi^2$ each. No reversibility assumption is needed.

Lemma 2.1. *If $f(a) = 0$, then*

$$\mathcal{E}_P(f, f) \geq \lambda \|f\|_\pi^2, \quad \lambda := \frac{1}{4m(n-1)}.$$

Proof. For each x , choose a simple path in the underlying undirected graph from x to a . It has length at most $n-1$. For each step choose one arc of G , ignoring its orientation. Telescoping along the path and applying Cauchy–Schwarz give

$$|f(x)|^2 \leq (n-1) \sum_{v, w} a_{vw} (f(v) - f(w))^2 = 4m(n-1) \mathcal{E}_P(f, f).$$

Averaging over x with the probability weights $\pi(x)$ proves the assertion. Each individual path sum was bounded by the same full arc sum before averaging; no path-congestion factor is introduced. $\square$

Lemma 2.2. *For every $a \in V$,*

$$\|PD_a\|_{L^2(\pi) \rightarrow L^2(\pi)} \leq q := \sqrt{1 - \lambda} < 1.$$

Proof. Jensen's inequality and stationarity imply

$$\|Rf\|_\pi^2 \leq \sum_{x,y} \pi(x) R(x,y) f(y)^2 = \|f\|_\pi^2.$$

The prescribed half-laziness therefore gives

$$\begin{aligned} \|Pf\|_\pi^2 &= \frac{1}{4}(\|f\|_\pi^2 + 2\langle f, Rf \rangle_\pi + \|Rf\|_\pi^2) \\ &\leq \frac{1}{2}(\|f\|_\pi^2 + \langle f, Rf \rangle_\pi) = \langle f, Pf \rangle_\pi = \|f\|_\pi^2 - \mathcal{E}_P(f, f). \end{aligned} \tag{5}$$

Apply this to $f = D_ag$, which vanishes at a , and use Lemma 2.1:

$$\|PD_ag\|_\pi^2 \leq (1 - \lambda)\|D_ag\|_\pi^2 \leq (1 - \lambda)\|g\|_\pi^2.$$

□

The order PD_a is useful here: the input function vanishes at the target before P acts. The argument neither takes a self-adjoint square root of P nor presupposes a real spectrum. Oliveira and Peres [2, Theorem 5] use products of killed operators to study moving targets for reversible chains with nonnegative spectrum. The representation of avoidance by such products is the same established method; Lemma 2.2 supplies the nonreversible estimate needed here through half-laziness and the Eulerian point inequality.

3 Survival and burn-in

Lemma 3.1. *Suppose μ is a nonnegative subprobability measure on V and $\mu(y) \leq L\pi(y)$ for all y , where $L \geq 1$. For any $v_1, \dots, v_k \in V$,*

$$\mu PD_{v_1} \cdots PD_{v_k} \mathbf{1} \leq \sqrt{L} q^k \leq L q^k.$$

The expression on the left is the mass surviving avoidance of v_i at time i , starting with the unnormalized measure μ .

Proof. The density $h(y) = \mu(y)/\pi(y)$ obeys

$$\|h\|_\pi^2 = \sum_y \frac{\mu(y)^2}{\pi(y)} \leq L \sum_y \mu(y) \leq L.$$

Using row measures in the matrix expression and column functions in the inner product, the surviving mass equals

$$\langle h, PD_{v_1} \cdots PD_{v_k} \mathbf{1} \rangle_\pi.$$

Cauchy–Schwarz, submultiplicativity of operator norms, Lemma 2.2, and $\|\mathbf{1}\|_\pi = 1$ bound this by $\sqrt{L} q^k$. This also covers $k = 0$. □

Proof of Theorem 1.1. At time T from (1), define the killed measure

$$\mu_T(y) = \mathbb{P}_x(X_T = y, \tau_u > T).$$

It is a subprobability and satisfies the pointwise bound

$$0 \leq \mu_T(y) \leq P^T(x, y) \leq \frac{5}{4}\pi(y).$$

Apply Lemma 3.1 with $L = 5/4$ to the suffix $u_{T+1}, \dots, u_{T+k}$. By the Markov property,

$$\mathbb{P}_x(\tau_u > T + k) \leq \frac{5}{4}q^k. \quad (6)$$

Since $\sqrt{1-\lambda} \leq e^{-\lambda/2}$, this proves (3). Summing the tail probabilities gives

$$\begin{aligned} \mathbb{E}_x \tau_u &= \sum_{t \geq 0} \mathbb{P}_x(\tau_u > t) \\ &\leq T + \frac{5}{4(1-q)} \leq T + \frac{5}{2\lambda} = T + 10m(n-1). \end{aligned}$$

Here $1-q = \lambda/(1+q) \geq \lambda/2$. The bound $T \leq c_1 mn$ from [1, Theorem 1.2] finishes the proof. $\square$

Keeping μ_T unnormalized is essential. We bound it by the unconditioned law at time T , rather than asserting that the law conditioned on avoiding the moving target is mixed. The latter assertion need not hold.

Corollary 3.2. *If $Y = (Y_t)_{t \geq 0}$ is any random vertex sequence independent of X , then (2) and (3) hold for $\tau = \inf\{t \geq 0 : X_t = Y_t\}$. In particular, they hold for the meeting time of two independent half-lazy simple walks on G , from any prescribed starting vertices.*

Proof. Condition on the entire trajectory of Y . The conditional law of X remains unchanged and the target sequence is deterministic. The bounds of Theorem 1.1 are uniform in that sequence, so integration gives the conclusion. $\square$

Independence cannot simply be omitted from this conditioning argument. For example, two walks on a directed cycle using identical increments preserve a nonzero displacement and never meet. Nor can half-laziness be discarded without replacing the proof: for a nonlazy walk on an even directed cycle, a deterministic target with the same velocity and a nonzero offset is never hit.

4 The coefficient-one distinction

The $O(mn)$ conclusion is not the numerical inequality $\mathbb{E}_x \tau_u \leq mn$. Even among simple, loopless, genuinely nonreversible Eulerian digraphs, the latter fails for a fixed target.

Proposition 4.1. *There is a connected simple loopless Eulerian nonreversible digraph with $n = 18$ and $m = 51$ for which the half-lazy simple walk has $\mathbb{E}_0 \tau_{17} = 936 > 918 = mn$.*

Proof. Start with the six vertices $0, 1, \dots, 5$ and all ordered nonloop arcs between them. Delete $2 \rightarrow 1$, $3 \rightarrow 2$, and $1 \rightarrow 3$, retaining the oppositely oriented triangle $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$. Attach a bidirected path of length 12, with

$$v_0 = 0, \quad v_k = 5 + k \quad (1 \leq k \leq 12).$$

There are 27 core arcs and 24 path arcs, hence $m = 51$ and $n = 18$. Every vertex has equal indegree and outdegree. The one-way triangle arcs prevent detailed balance.

For a fixed target at $17 = v_{12}$, set

$$\begin{aligned} H(0) &= 936, & H(1) &= H(2) = H(3) = \frac{2834}{3}, & H(4) &= H(5) = 945, \\ H(v_k) &= 936 - 54k - 2k^2 & & & (1 \leq k \leq 12). \end{aligned}$$

In particular, $H(17) = 0$. The half-lazy first-step equations at each $x \neq 17$ are equivalent to

$$\sum_{y: a_{xy}=1} (H(x) - H(y)) = 2d(x). \quad (7)$$

At vertex 0, the left side is $56 - 26 - 18 = 12 = 2d(0)$. At each of 1, 2, 3, it is $26/3 - 2/3 = 8 = 2d(x)$. At each of 4, 5, it is $9 + 3(1/3) = 10 = 2d(x)$. At an internal path vertex the quadratic second difference is $4 = 2d(x)$. Thus H solves all the first-step equations with zero boundary value at 17. Their unique solution on this finite irreducible chain is the expected hitting time, proving the assertion. $\square$

This is a dense-core-and-tail construction, not a claim of a new counterexample method. It concerns a fixed target, which is allowed among arbitrary deterministic trajectories but is not an independent copy of the same walk. Consequently it does not disprove the coefficient-one inequality for two independent half-lazy walks.

Thus the completed assertion is the logarithm-removal question in [1], not every interpretation of the coefficient-one formulation in [3].

5 Reproducibility

The source archive includes the analytic proof and two exact-rational Python checkers. One checks the energy identity, killed operators and survival on 44 fixtures, including 204 positive-semidefiniteness certificates, 34 520 target trajectories and 824 comparisons with direct path enumeration. The other solves the 17 hitting-time equations of Proposition 4.1 by rational Gaussian elimination and checks all 18 displayed values. Both were run with and without Python optimization. These finite regression tests supplement, but do not replace, the analytic proof.

This preprint is unrefereed; no independent review, formal verification or absolute priority is claimed. AI assistance was used for mathematical exploration, drafting and reproducibility checks. The author is responsible for the manuscript and its claims.

References

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