

Constant Normalized Area in the Planar Three-Body Problem

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Abstract

We prove that a collision-free noncollinear planar Newtonian three-body motion with constant triangle area divided by barycentric moment of inertia is necessarily a Lagrange homographic motion. The masses are arbitrary positive numbers, and the conclusion holds on any open time interval. Consequently, the collision-free exactly periodic planar solutions with constant unsigned normalized area are precisely the elliptic Lagrange and Euler families, including their circular members. In mass-Jacobi shape coordinates the area condition fixes a latitude. We continue necessary longitude, normal-force and energy identities to an isolated complex binary branch point. Two possible vanishing orders exhaust the cases and give incompatible leading energy coefficients. The reduction and classical families are standard; the proof uses an analytic energy-comparison method related to work on Saari's conjecture. The result addresses the planar part of an AIM workshop question, not its genuinely spatial extension.

1 Statement and relation to the workshop question

Let $m_1, m_2, m_3 > 0$ and $G > 0$. In the center-of-mass frame consider a classical collision-free solution of

$$\ddot{q}_i = G \sum_{j \neq i} m_j \frac{q_j - q_i}{|q_j - q_i|^3}, \quad q_i \in \mathbb{R}^2, \quad \sum_i m_i q_i = 0. \quad (1)$$

Write $M = m_1 + m_2 + m_3$ and

$$I = \sum_i m_i |q_i|^2, \quad \Delta = \frac{1}{2} \det(q_2 - q_1, q_3 - q_1).$$

Unsigned area is $|\Delta|$. A nonzero constant unsigned ratio has fixed sign on a connected interval, so using signed area loses no such solution.

Theorem 1.1. *If a noncollinear solution of (1) has constant Δ/I on an open interval, then its shape is constant and its triangle is equilateral. The motion is Lagrange homographic.*

Corollary 1.2. *The collision-free exactly periodic planar Newtonian three-body solutions with constant $|\Delta|/I$ are precisely the elliptic Lagrange and elliptic Euler homographic families, including circles. Their respective ratios are*

$$\frac{\sqrt{3} M}{4(m_1 m_2 + m_2 m_3 + m_3 m_1)} \quad \text{and} \quad 0.$$

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Periodicity here means periodicity of the labelled positions in an inertial frame, not merely closure modulo a rotation. A periodic center of mass is constant, justifying the chosen frame. No collision regularization is used.

Problem 18 of the AIM workshop list [1], attributed to Wu-Yi Hsiang, asks for periodic planar or spatial motions with this constant ratio. Our result classifies the planar solutions; a triangle whose plane precesses in space is not covered. The source's separate remark about meridians is also not answered here.

The mass-triangle reduction and Newtonian shape potential are standard; we use the framework of Hsiang–Straume [2]. The classical homographic results are surveyed by Chenciner [4]. Analytic continuation of an energy necessary condition followed by a Puiseux coefficient comparison appears in Fujiwara–Fukuda–Ozaki–Taniguchi [3, Section 5]. Their invariant is $\sqrt{I}U$, and their theorem concerns equal masses. Neither assumption is substituted for the normalized-area condition used here. The argument below instead uses the isolated complex binary singularities of a latitude.

2 Area coordinates and necessary equations

Set $m_{12} = m_1 + m_2$, $\mu_1 = m_1 m_2 / m_{12}$ and $\mu_2 = m_3 m_{12} / M$. Identify the physical plane with $\mathbb{C}$ and use

$$x = \sqrt{\mu_1}(q_2 - q_1), \quad y = \sqrt{\mu_2} \left(q_3 - \frac{m_1 q_1 + m_2 q_2}{m_{12}} \right).$$

Then $I = |x|^2 + |y|^2 = r^2$ and

$$a = 4\sqrt{m_1 m_2 m_3 / M} \frac{\Delta}{I} = \frac{2 \operatorname{Im}(\bar{x}y)}{r^2}, \quad |a| \leq 1. \quad (2)$$

Reflection permits $a > 0$. At $a = 1$, equality in $2|\operatorname{Im}(\bar{x}y)| \leq 2|x||y| \leq |x|^2 + |y|^2$ fixes the normalized Gram matrix and thus the shape. For now suppose $0 < a < 1$ and put

$$c = \cos \chi = a, \quad s = \sin \chi \in (0, 1).$$

The unitary transformation $z_1 = (x - iy)/\sqrt{2}$, $z_2 = (x + iy)/\sqrt{2}$ allows local coordinates

$$z_1 = r \cos(\chi/2) e^{i(\alpha+\beta/2)}, \quad z_2 = r \sin(\chi/2) e^{i(\alpha-\beta/2)}.$$

Here α is physical rotation. Direct differentiation gives

$$T = \frac{\dot{r}^2}{2} + \frac{r^2 \dot{\chi}^2}{8} + \frac{r^2}{2} \left(\dot{\alpha}^2 + \frac{\dot{\beta}^2}{4} + \cos \chi \dot{\alpha} \dot{\beta} \right). \quad (3)$$

The three mutual differences are real linear combinations $u_j x + v_j y$ with distinct coefficient lines. Indeed their vectors are

$$(\mu_1^{-1/2}, 0), \quad \left(\frac{m_2}{m_{12}\sqrt{\mu_1}}, \frac{1}{\sqrt{\mu_2}} \right), \quad \left(-\frac{m_1}{m_{12}\sqrt{\mu_1}}, \frac{1}{\sqrt{\mu_2}} \right).$$

Every pair has nonzero determinant. Using

$$\frac{|x|^2 - |y|^2}{r^2} = s \cos \beta, \quad \frac{2 \operatorname{Re}(\bar{x}y)}{r^2} = -s \sin \beta,$$

we obtain, for each pair,

$$\frac{|u_j x + v_j y|^2}{r^2} = k_j [1 - s \cos(\beta - \phi_j)], \quad (4)$$

$$k_j = (u_j^2 + v_j^2)/2, \quad (\cos \phi_j, \sin \phi_j) = \frac{(v_j^2 - u_j^2, 2u_j v_j)}{u_j^2 + v_j^2}. \quad (5)$$

Thus $k_j > 0$ and the three ϕ_j are distinct modulo 2π . For positive constants b_j incorporating G and the masses,

$$U = \frac{V}{r}, \quad V(\chi, \beta) = \sum_{j=1}^3 b_j [1 - \sin \chi \cos(\beta - \phi_j)]^{-1/2}. \quad (6)$$

All brackets are positive for real β and $0 < s < 1$. This is the binary-factor potential of [2, Eq. (35)], with the present longitude convention.

Let $\ell = r^2(\dot{\alpha} + c\dot{\beta}/2)$ be the conserved scalar angular momentum and set $w = r^2\dot{\beta}$. Vary χ before setting it constant. The Euler–Lagrange equations for $T + U$ and the energy give

$$\frac{s^2}{4}\dot{w} = \frac{V_\beta}{r}, \quad rV_\chi = \frac{s}{4}w(2\ell - cw), \quad (7)$$

$$H = \frac{\dot{r}^2}{2} + \frac{\ell^2 + s^2 w^2/4}{2r^2} - \frac{V}{r}. \quad (8)$$

In particular, the second equation of (7) is a normal-force constraint, not an equation obtained by constraining the motion by hand.

Suppose the shape is nonconstant. Real analyticity supplies an interval with $w \neq 0$ on which β is a coordinate. The singularity calculation below shows that V_χ is not identically zero. Its real zeros are isolated, so shorten the interval to ensure $V_\chi \neq 0$. With $Y = 2\ell - cw$ and $Q = V_\beta/V_\chi$, equations (7) imply

$$w\beta = \frac{YQ}{s}, \quad Y_\beta = -\frac{cQ}{s}Y, \quad r = \frac{swY}{4V_\chi}. \quad (9)$$

Here $Y \neq 0$ on the interval. Since $\dot{r} = r_\beta w/r^2$, clearing denominators in (8) yields

$$2Hr^4 = w^2 r_\beta^2 + (\ell^2 + s^2 w^2/4)r^2 - 2Vr^3. \quad (10)$$

3 The complex binary obstruction

Choose any j and let $\beta_* = \phi_j + i\eta$, where $\cosh \eta = 1/s$ and $\sinh \eta = c/s$. For $f = 1 - s \cos(\beta - \phi_j)$ one has

$$f(\beta_*) = 0, \quad f_\beta(\beta_*) = ic, \quad f_\chi(\beta_*) = -c/s. \quad (11)$$

The zero is simple. The other two binary factors have different real parts of their zeros modulo 2π , so they are nonzero at β_* . In particular the singular contribution of the j th summand to V_χ cannot cancel.

Lemma 3.1. *The germs in (9) can be continued to a punctured neighborhood of a point over β_* so that, with $u^2 = \beta - \beta_*$,*

$$V = v/u + O(1), \quad V_\chi = d/u^3 + O(u^{-1}), \quad Q = -is + O(u^2), \quad vd \neq 0. \quad (12)$$

The continued Y is holomorphic at $u = 0$ and has $Y(0) \neq 0$.

Proof. With $z = e^{i\beta}$ the distance factors are rational functions of z . Adjoining their square roots gives a finite algebraic function field and a connected compact Riemann surface for the physical germ. The functions V, V_β, V_χ and Q are meromorphic there. Remove the points over $z = 0, \infty$, the other binary points and the poles of Q . The relevant exceptional set is finite, and its removal does not disconnect the surface. A path from the physical germ reaches a punctured neighborhood of a point over the selected binary zero. The linear Y equation continues along it.

At the endpoint, only the j th summand is singular. The simple zero in (11) gives the first two expansions in (12), while V_β/V_χ has the limit $f_\beta/f_\chi = -is$. The other summands are holomorphic in β , so Q is holomorphic in u and has the stated error. Square-root signs do not affect this ratio.

In this coordinate the remaining equation is

$$\frac{dY}{du} = -\frac{2cu}{s}Q(u)Y.$$

Its coefficient is holomorphic at zero. The local exponential solution is finite and nonzero there because the continued solution was nonzero. $\square$

The energy identity (10) continues along the same path. This is continuation of necessary analytic identities, not a claim that a real physical trajectory reaches a complex collision. In particular its squares remain algebraic squares, not absolute squares.

Put $w_0 = w(0)$ and $Y_0 = Y(0)$. Since $c \neq 0$, $w = (2\ell - Y)/c$ is also holomorphic. If $w_0 \neq 0$, then

$$r = Ru^3 + O(u^5), \quad r_\beta = \frac{3}{2}Ru + O(u^3), \quad R = \frac{sw_0Y_0}{4d} \neq 0.$$

The right side of (10) has the nonzero leading coefficient $\frac{9}{4}w_0^2R^2u^2$ from its radial term. Its other terms have orders at least 6 and 8, and the left side has order at least 12. This is impossible.

If $w_0 = 0$, then $Y_0 = 2\ell \neq 0$. Equations (9) and (12) yield

$$w = -2i\ell u^2 + O(u^4), \quad r = Ru^5 + O(u^7), \quad r_\beta = \frac{5}{2}Ru^3 + O(u^5), \quad R \neq 0.$$

The combined order- u^{10} coefficient of the radial and $\ell^2 r^2$ terms is

$$\left(\frac{25}{4}(-2i\ell)^2 + \ell^2\right)R^2 = -24\ell^2R^2 \neq 0. \quad (13)$$

The remaining kinetic and potential terms have order at least 14, and the left side has order at least 20. This is again impossible. The conjugate binary point reverses i but leaves both contradictions unchanged.

These alternatives exhaust the possibilities, excluding every nonconstant noncollinear latitude arc. The endpoint $a = 1$ already has constant shape. This proves the constant-shape part of Theorem 1.1 throughout the connected interval.

4 Homographic identification and periodic solutions

For a fixed noncollinear planar shape write $q_i(t) = \zeta(t)Q_i$, with fixed $Q_i \in \mathbb{C}$. Equation (1) gives

$$\zeta''Q_i = \frac{\zeta}{|\zeta|^3}F_i(Q).$$

Hence $F_i(Q) = \lambda Q_i$ for a common constant complex scalar λ . Zero total torque makes λ real, and the virial identity makes it negative. Write $\lambda = -\kappa$, $\kappa > 0$. The central equations become

$$\sum_{j \neq i} m_j \left(\frac{G}{|Q_j - Q_i|^3} - \frac{\kappa}{M} \right) (Q_j - Q_i) = 0.$$

The two edge vectors at a noncollinear vertex are independent. Their coefficients vanish, and all three side lengths are equal. The scalar equation is Kepler's equation. This completes Theorem 1.1.

For zero area write $q_i = s_i(t)u(\theta(t))$, where u is a planar unit vector. If $\ell \neq 0$, then $\dot{\theta} = \ell/I$ never vanishes. The components normal to the line give

$$2\dot{s}_i\dot{\theta} + s_i\ddot{\theta} = 0 \quad \text{for every } i.$$

All s_i have a common dilation factor, including any identically zero coordinate, so the shape is fixed and is an Euler central configuration. If $\ell = 0$, the line is fixed. Before a collision the body ordering is fixed, and the distance between the two outer bodies has strictly negative second derivative. It cannot be a positive periodic function.

Kepler's classification now shows that all collision-free periodic cases are elliptic Lagrange or Euler motions. Conversely these families have constant normalized area. An equilateral triangle of side d has $|\Delta| = \sqrt{3}d^2/4$ and $I = d^2(m_1m_2 + m_2m_3 + m_3m_1)/M$, giving the stated ratio. Euler configurations have zero area. This proves Corollary 1.2.

5 Spatial limitation and reproducibility

The fact that three points always form a triangle does not make every spatial motion planar. To see the missing variables, write a spatial mass-Jacobi matrix as $A = RDQ(\beta/2)$, where $D = r \operatorname{diag}_{3 \times 2}(\cos k, \sin k)$, $c = \sin 2k$ and $s = \cos 2k$. If L_1, L_2, L_3 are the components of conserved spatial angular momentum in the moving principal frame, the normal equation at constant area is

$$rV_\chi = \frac{s}{4}w(2L_3 - cw) + c \left(\frac{L_1^2}{(1-s)^2} - \frac{L_2^2}{(1+s)^2} \right). \quad (14)$$

The longitudinal momentum is $cL_3/2 + s^2w/4$, and L_3 is generally not constant. Thus the scalar linear equation in (9), essential to Lemma 3.1, is unavailable. A real angular-momentum bound does not bound its complex components after continuation. We do not claim a classification of such precessing motions.

The accompanying source archive contains a reproducible SymPy calculation of 26 exact identities: the Cartesian kinetic reduction, both planar and spatial normal constraints, angular momenta, and the two leading energy coefficients. Normal and optimized runs agree. These are algebraic checks, not a formal verification of the analytic proof.

This unrefereed preprint was prepared with AI assistance in derivation, literature comparison, symbolic checks and exposition. It has been self-audited, not independently refereed. No absolute-priority claim is made.

References

- [1] American Institute of Mathematics, *Variational Methods in Celestial Mechanics: Problems*, Problem 18, attributed to Wu-Yi Hsiang, p. 6. <https://aimath.org/WWN/varcelest/varcelest.pdf>.
- [2] W.-Y. Hsiang and E. Straume, *The planar 3-body problem II: reduction to pure shape and spherical geometry (2nd version)*, arXiv:1803.06866 (2018). <https://arxiv.org/abs/1803.06866>.
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- [4] A. Chenciner, *The Lagrange reduction of the N-body problem, a survey*, arXiv:1111.1334 (2011). <https://arxiv.org/abs/1111.1334>.