

More Colors Can Slow Mixing on a Four-Vertex Path

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Abstract

Consider the discrete-time coloring chain that chooses a uniform vertex and a uniform color from all q colors, accepting the proposed recoloring when it is proper and otherwise staying in place. On the four-vertex path, the worst-start total-variation mixing time at threshold $1/e$ equals 8 for $q = 17$ and 9 for $q = 18$. Thus increasing the number of colors can increase the mixing time even on a connected tree with no isolated vertices. We give an exact finite certificate, verified both by full-state integer enumeration and by a color-symmetry quotient. An elementary comparison with independent coordinate refresh also proves that the mixing time is 9 for every $q \geq 512$. The result concerns the literal proposal chain and a fixed finite graph, not an asymptotic slowdown on growing lattices.

1 The chain and the result

Let $G = (V, E)$ be a finite simple graph, with $n = |V|$, $m = |E|$ and maximum degree Δ . Write $\Omega_{G,q}$ for its proper colorings with colors $\{0, \dots, q-1\}$. At each step, independently choose $v \in V$ uniformly and $c \in \{0, \dots, q-1\}$ uniformly. Replace the color at v by c if the result is proper, and otherwise leave the coloring unchanged. The current color is among the possible proposals. Every rejected proposal consumes one step.

The transition matrix $P_{G,q}$ is symmetric. Indeed, a nontrivial one-vertex recoloring and its reverse both have probability $1/(nq)$. Consequently the uniform law $\pi_{G,q}$ on $\Omega_{G,q}$ is stationary. Set

$$d_{G,q}(t) = \max_{a \in \Omega_{G,q}} \|P_{G,q}^t(a, \cdot) - \pi_{G,q}\|_{\text{TV}}, \quad t_{G,q} = \min\{t \in \mathbb{Z}_{\geq 0} : d_{G,q}(t) \leq e^{-1}\}.$$

The cases considered below satisfy $q \geq \Delta + 2$.

Question 5 of the AIM *Phase Transitions* problem list, compiled by Charles Radin, asks whether this mixing time decreases as the number of colors increases [1]. A frozen upstream report associated with record AIM-PHYSICS-0025 already gives a negative answer using two isolated vertices [3]. We credit that elementary example and address the restriction to connected graphs.

Theorem 1. *For the path P_4 on four vertices,*

$$t_{P_4,17} = 8 < 9 = t_{P_4,18}.$$

Moreover, $t_{P_4,q} = 9$ for every integer $q \geq 512$.

The first assertion is a finite computer-assisted result. Its certificate consists of integer recurrences and exact rational comparisons given below. The second assertion follows from an analytic comparison. Neither a minimal graph nor an optimal color pair or onset is asserted.

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The result gives a connected counterexample to the finite-graph monotonicity question. It does not answer the separate square-lattice rate discussion or the other questions appended to the frozen dataset record.

The normalization is essential. Choosing uniformly among legal colors, using continuous time, counting sweeps, or inserting an extra holding probability gives a different chain. In particular, the legal-color heat-bath chain described in [2, Section 3.3] is not substituted for the proposal chain used here.

2 An exact full-state certificate

Label the vertices of P_4 in path order 1, 2, 3, 4. There are

$$Z_q = q(q-1)^3$$

proper colorings. For an initial coloring a , let $N_t(x)$ count sequences of t vertex-color proposals taking a to x . There are $(4q)^t$ such proposal sequences in total, so

$$N_0(x) = \mathbf{1}_{\{x=a\}}, \quad P_{P_4,q}^t(a, x) = \frac{N_t(x)}{(4q)^t}.$$

For a proper coloring x , let $A_v(x)$ be the colors not used by its neighbors, and write $x^{v \leftarrow c}$ for the coloring obtained by replacing its v -th coordinate. The total number of forbidden vertex-color proposals at x is

$$b(x) = 4 + \mathbf{1}_{\{x_1 \neq x_3\}} + \mathbf{1}_{\{x_2 \neq x_4\}}.$$

The two endpoints each forbid one color. Each interior vertex forbids one or two colors according as its neighbors agree or disagree.

Lemma 2. *The exact incoming-count recurrence is*

$$N_{t+1}(x) = b(x)N_t(x) + \sum_{v=1}^4 \sum_{c \in A_v(x)} N_t(x^{v \leftarrow c}). \quad (1)$$

If $B = (4q)^t$ and $S = \{x : Z_q N_t(x) > B\}$, then

$$\|P_{P_4,q}^t(a, \cdot) - \pi_{P_4,q}\|_{\text{TV}} = \frac{Z_q \sum_{x \in S} N_t(x) - |S|B}{Z_q B}. \quad (2)$$

Proof. A rejected proposal contributes a self-loop, giving the first term in (1). For an accepted proposal at v ending at x , the previous coloring must be $x^{v \leftarrow c}$ with $c \in A_v(x)$. It contributes one incoming proposal. The choice $c = x_v$ is included: this accepted self-proposal is distinct from each forbidden proposal. This counts every possibility exactly once. Formula (2) sums the positive part of $N_t(x)/B - 1/Z_q$, which equals the total variation distance. $\square$

Global permutations of color names preserve both transition probabilities and the uniform stationary law. Every proper initial coloring is equivalent to exactly one of the five equality patterns

$$0101, \quad 0102, \quad 0120, \quad 0121, \quad 0123.$$

Thus computing these five initial states is sufficient for the maximum over all 69 632 proper colorings at $q = 17$, or all 88 434 proper colorings at $q = 18$.

Table 1 gives exact rational enclosures from (1)–(2). An entry M means that the corresponding distance D satisfies $M/10^8 \leq D < (M+1)/10^8$; it is a certified integer floor, not a floating-point approximation.

Initial pattern	$q = 17, t = 8$	$q = 18, t = 8$	$q = 18, t = 9$
0101	36360587	36427175	29047344
0102	36546172	36606107	29188200
0120	36750813	36802419	29343553
0121	36546172	36606107	29188200
0123	36740364	36792860	29336313

Table 1: Exact enclosures of the total-variation distances.

In particular the decisive maxima are

$$d_{P_4,17}(7) = \frac{1526270343323}{3361494409216}, \quad d_{P_4,17}(8) = \frac{8549318601}{23262937088}, \quad (3)$$

$$d_{P_4,18}(8) = \frac{326454845236864469}{887047230421204992}, \quad d_{P_4,18}(9) = \frac{3123494132369989291}{10644566765054459904}. \quad (4)$$

For a rational threshold certificate, put

$$L = \sum_{j=0}^{22} \frac{1}{j!}, \quad U = L + \frac{24}{23 \cdot 23!}.$$

The positive tail of the exponential series is strictly smaller than the geometric majorant $24/(23 \cdot 23!)$, so $L < e < U$. Integer comparison after taking reciprocals gives

$$0.36787944117 < e^{-1} < 0.36787944118.$$

Equations (3)–(4) and the table imply

$$d_{P_4,17}(7) > e^{-1} > d_{P_4,17}(8), \quad d_{P_4,18}(8) > e^{-1} > d_{P_4,18}(9).$$

Total variation contracts under a Markov kernel, as follows directly by applying the triangle inequality to $\sum_x (\mu(x) - \pi(x))P(x, y)$ and summing over y . Thus $d_{G,q}(t)$ is nonincreasing in t , and these comparisons prove the first assertion of Theorem 1.

3 Reproducible computation and a second representation

The supplementary source archive contains all code and exact output. No random sampling or floating-point threshold comparison is used. There are two structurally different implementations.

3.1 Full-state integer arrays

The direct implementation stores N_t in a $q \times q \times q \times q$ array, with zero entries at improper colorings. For a proper destination x , the inner sum in (1) is the sum along coordinate v : improper predecessor entries are already zero. It therefore forms the four axis sums, masks the result to proper destinations, and adds $b(x)N_t(x)$. It checks total mass $(4q)^t$ and zero mass at every improper coloring at every step.

For $q = 17, 18$ and $0 \leq t \leq 9$, signed 64-bit arrays are safe because $(4q)^9 < 2^{63} - 1$. Every nonnegative intermediate count and axis sum is at most the total proposal count for that step. Before the products in (2), the array sums and cardinalities are converted to arbitrary-precision Python integers. Fractions are then reduced exactly.

3.2 A color-symmetry quotient

A separate implementation fixes by name the r colors appearing in the initial state and quotients by permutations of the remaining $q - r$ colors. It retains the equality pattern of every fresh color, naming fresh classes by order of first occurrence. A representative using k fresh classes has orbit weight

$$w = (q - r)_k,$$

where $(s)_k = s(s - 1) \cdots (s - k + 1)$. Each fixed or currently represented fresh color has proposal multiplicity one; a new fresh color has multiplicity $q - r - k$. After acceptance or rejection the state is canonically renamed. Summing proposal multiplicities gives an integer matrix C with every row sum $4q$.

For the five initial patterns, this yields respectively 67, 188, 188, 188, 449 orbit states. The implementation checks the exact balance identities

$$w_i C_{ij} = w_j C_{ji}.$$

Starting from the initial coloring, the distribution remains constant inside each orbit because the chain commutes with permutations of unused colors. The stationary orbit mass is w_i/Z_q . Consequently the quotient computes the full total variation exactly; it is not merely a projection giving a lower bound.

The quotient uses arbitrary-precision integers throughout and does not depend on the tensor implementation. All 100 exact distances (two color numbers, five initial patterns, ten times) agree between the two methods. Each method also passes both normal Python execution and optimized execution. The portable checker additionally verifies 261 edgeless consistency cases against the formula in Section 5. These tests are supporting implementation checks; the finite certificate is the stated recurrence over the full enumerated state spaces.

From the archive's `reproducibility` directory, run

```
python3 -B check_certificate.py fresh-certificate.json
python3 -O -B check_certificate.py fresh-optimized.json
python3 -B check_full_tensor.py fresh-full-state.json
```

The first two commands use only the Python standard library. The third requires NumPy. Each command refuses to overwrite an existing receipt. This is a computer-assisted proof, not proof-assistant formalization or independent expert review.

4 A comparison valid for every fixed graph

Let $C_n(t)$ be the probability that t independent uniform choices from n vertices have not visited every vertex.

Proposition 3. *For a finite simple graph with n vertices, m edges and maximum degree Δ , whenever $\Omega_{G,q}$ is nonempty,*

$$C_n(t) - \frac{n}{q} \leq d_{G,q}(t) \leq C_n(t) + \frac{t\Delta + m}{q}. \quad (5)$$

In particular $d_{G,q}(t) \rightarrow C_n(t)$ as $q \rightarrow \infty$ for fixed G, t .

Proof. Fix an initial coloring a , and let A be the event that at least one vertex has its initial color. If some vertex has not been selected, then A occurs, so the time- t probability of A is at least $C_n(t)$. Color-permutation symmetry makes each stationary vertex marginal uniform on the q colors. Hence $\pi_{G,q}(A) \leq n/q$. The event characterization of total variation gives the lower bound.

For the upper bound, start both the proper-coloring chain and the unconstrained refresh chain at a , and use the same vertex and color proposals. Until their first discrepancy, a proposal is forbidden with probability at most Δ/q . A union bound gives distance at most $t\Delta/q$ between their laws at time t .

The refresh chain has the uniform product law ν_q as stationary law. Couple it to a stationary refresh chain with the same proposals. After all coordinates have been selected, the two chains agree. The usual coordinate-coupling argument [2, Chapter 5] bounds its distance to ν_q by $C_n(t)$. Finally, $\pi_{G,q}$ is ν_q conditioned on properness. Its distance to ν_q equals the probability of an improper product coloring, which is at most m/q by the union bound on edges. The triangle inequality proves the upper bound. $\square$

For P_4 , inclusion-exclusion yields

$$C_4(t) = \sum_{j=1}^4 (-1)^{j+1} \binom{4}{j} \left(\frac{4-j}{4}\right)^t, \quad C_4(8) = \frac{3089}{8192}, \quad C_4(9) = \frac{4729}{16384}.$$

Since $\Delta = 2$ and $m = 3$, for every $q \geq 512$ equation (5) gives

$$d_{P_4,q}(8) \geq \frac{3025}{8192} > e^{-1}, \quad d_{P_4,q}(9) \leq \frac{5401}{16384} < e^{-1}.$$

This proves the second assertion of Theorem 1. More generally, if a threshold ε lies strictly between $C_n(T)$ and $C_n(T-1)$, the corresponding mixing time of every fixed graph equals T for all sufficiently large q . The bound 512 above is convenient, not sharp.

5 Prior example and scope

The upstream report [3] already observes that for the edgeless two-vertex graph E_2 ,

$$d_{E_2,q}(t) = 2^{1-t}(1 - 1/q)^2, \quad t \geq 1.$$

Indeed, after at least one step the probability that a specified coordinate has never been refreshed is 2^{-t} , and the two untouched-coordinate events are disjoint. Otherwise both coordinates have their independent uniform colors. The signed difference from the uniform product law is positive precisely on states with at least one initial-color match (up to zero differences when $q = 2$); summing its positive part gives the displayed formula. Therefore $t_{E_2,2} = 1 < t_{E_2,3} = 2$.

That observation is prior context, not a contribution claimed here. The present certificate shows that removing isolated vertices does not restore monotonicity: a connected four-vertex tree already suffices. The comparison with the coupon collector uses standard coupling ideas; its explicit application is proved in full.

The increase $8 \rightarrow 9$ is a fixed-threshold, finite-state phenomenon. No growing-graph asymptotic slowdown, spectral-gap monotonicity theorem, or statement for a differently normalized coloring dynamics follows. The bounded literature check found no identical connected-path certificate, but no absolute priority claim is made. This preprint is unrefereed.

AI assistance was used for mathematical exploration, exact-checking code, source comparison and exposition. The originating research workflow includes a detailed self-audit and the reproducible computations described above; it does not constitute independent peer review.

References

- [1] American Institute of Mathematics, *Phase Transitions: problem list*, compiled by Charles Radin, Question 5, pp. 1–2, undated.
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- [2] D. A. Levin and Y. Peres, with contributions by E. L. Wilmer, *Markov Chains and Mixing Times*, second edition, American Mathematical Society, 2017.
<https://pages.uoregon.edu/dlevin/MARKOV/mcmt2e.pdf>.
- [3] `ulamai/UnsolvedMath`, version 1.6.0, record AIM-PHYSICS-0025 and its accompanying research report, frozen dataset commit `c6e7f41b4eb6ce876feb83e7eb626be1b243fd7e`.
<https://huggingface.co/datasets/ulamai/UnsolvedMath>. The report supplies the edgeless example credited in Section 5.