

When Binary-Form Norm Balls Have Finite Haar Volume

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Abstract

For a nonzero real binary form of degree d , we show that all norm sublevel sets in $\mathrm{SL}_2(\mathbb{R})$ have finite Haar volume if and only if every real projective root has multiplicity strictly less than $d/2$. For the Bombieri norm, failure of this condition makes every nonempty sublevel set have infinite volume. The proof uses a positive-weight criterion for properness and an explicit thickening in Iwasawa coordinates. The cubic x^2y has trivial stabilizer but infinite-volume norm balls at every positive radius, in every norm. We compute its exact cutoff volume and give a critical quartic with stabilizer $\{I, -I\}$ whose small balls are empty. These calculations exhibit a normalization obstruction in the finite-stabilizer setup preceding AIM Question 19 on representation-ball averages; they do not determine quotient limits under corrected hypotheses.

1 The normalization question

Let a group G act linearly on a finite-dimensional normed real space V , and write

$$B_T(v) = \{g \in G : \|gv\| < T\}, \quad \Phi_v(g) = gv.$$

In the paragraph preceding Question 19 of the AIM measure-rigidity list [1, Section 4.3, p. 8], G is semisimple and v is assumed to have finite stabilizer. The text infers compactness of these sets and asks about the limit of their normalized Haar measures projected to G/Γ , where Γ is a lattice. The normalization itself requires $0 < m_G(B_T(v)) < \infty$.

Our subject is this prerequisite, not the subsequent equidistribution problem. The frozen research report associated with record **AIM-OTHER-0095** in [2] already identifies x^2y as a trivial-stabilizer vector with nonclosed orbit. It leaves the Haar-volume calculation open. We retain that attribution and supply the volume calculation, a binary-form finiteness criterion, and an exact cutoff formula. The strict-versus-closed sublevel distinction is not the obstruction: the example has infinite volume even with a strict inequality.

The properness condition is consistent with the framework of Gorodnik–Weiss [3, Section 1], which starts from a proper distance function on the group. A vector norm $\|gv\|$ need not have that property. Tanimoto–Tanis [4, Introduction and Section 1] study height counting for binary forms with distinct roots and use a compatible $\mathrm{SO}_2(\mathbb{R})$ -invariant norm. The repeated-root examples below do not satisfy their distinct-root hypothesis.

The arguments use standard facts about Haar measure, Cartan decomposition and representation weights. This unrefereed note was prepared with AI assistance and self-audited. No absolute priority or formal certification is claimed.

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2 Properness and Haar volume

The following packing observation is useful before making any representation-specific calculation.

Lemma 1. *Let G be a locally compact second countable unimodular group acting continuously and linearly on a finite-dimensional normed real space. Then Φ_v is proper if and only if $m_G(B_T(v)) < \infty$ for every finite $T > 0$. If Φ_v is not proper, all sufficiently large balls have infinite measure.*

Proof. If Φ_v is proper, the inverse image of a closed vector ball is compact and contains $B_T(v)$. Conversely, if Φ_v is not proper, for some $M < \infty$ the closed set $E = \{g : \|gv\| \leq M\}$ is not compact. Choose a compact identity neighborhood U with positive Haar measure, and let $C = \max_{u \in U} \|\rho(u)\|_{\text{op}}$. Inductively choose $g_n \in E$ outside $\bigcup_{i < n} U^{-1}Ug_i$. This is possible because the union is compact and E is not contained in a compact set. The sets Ug_n are disjoint. Unimodularity gives $m_G(Ug_n) = m_G(U) > 0$, whereas $\|ug_nv\| \leq CM$ for $u \in U$. Consequently $m_G(B_T(v)) = \infty$ whenever $T > CM$. $\square$

We now specialize to $G = \text{SL}_2(\mathbb{R})$ and $K = \text{SO}_2(\mathbb{R})$. On real degree- d binary forms use the action and norm

$$(\rho(g)f)(x, y) = f((x, y)g), \quad \|f\|_B^2 = \sum_{j=0}^d \binom{d}{j}^{-1} c_j^2, \quad f = \sum_{j=0}^d c_j x^j y^{d-j}. \quad (1)$$

This is the symmetric-tensor norm induced by the Euclidean norm on $\mathbb{R}^2$; in particular, it is K -invariant. The monomials are orthogonal. For a finite direct sum of these representations, use the orthogonal direct sum norm, also denoted $\|\cdot\|_B$.

Put

$$a_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \quad n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The a_r -weight of $x^j y^{d-j}$ is $2j - d$. Every group element has unique coordinates

$$g = k_\theta n_t a_r = \begin{pmatrix} r \cos \theta & (t \cos \theta - \sin \theta)/r \\ r \sin \theta & (t \sin \theta + \cos \theta)/r \end{pmatrix}, \quad r > 0, \quad t \in \mathbb{R}, \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}. \quad (2)$$

Indeed r and θ are determined by the first column; the determinant then determines the remaining transverse component. We normalize Haar measure by

$$dm_G = d\theta dt \frac{dr}{r}. \quad (3)$$

To check the normalization, on the chart $g = \begin{pmatrix} a & b \\ c & (1+bc)/a \end{pmatrix}$, $a \neq 0$, the invariant density is $da db dc/|a|$. Multiplication on either side by a fixed determinant-one matrix has chart Jacobian a'/a , where a' is the new first entry, proving invariance of the density. The Jacobian of $(r, t, \theta) \mapsto (a, b, c)$ in (2) is $\cos \theta$. Since $a = r \cos \theta$, this gives (3). The omitted chart boundary has measure zero.

Theorem 2. *Let V be a finite direct sum of real binary-form representations, with the norm just defined. Let P_+ be the orthogonal projection onto the strictly positive a_r -weight spaces, and set*

$$c(v) = \min_{k \in K} \|P_+ \rho(k)v\|_B.$$

Then Φ_v is proper if and only if $c(v) > 0$. If it is not proper, every nonempty Bombieri-norm ball $B_T(v)$ has infinite Haar measure.

Proof. The minimum exists by compactness. If $c(v) > 0$, there is a positive weight; let ℓ be the smallest one occurring in V . Cartan decomposition writes $g = k_1 a_r k_2$ with $r \geq 1$. Orthogonality and K -invariance give

$$\|\rho(g)v\|_B \geq r^\ell \|P_+\rho(k_2)v\|_B \geq c(v)r^\ell. \quad (4)$$

A bounded orbit norm therefore bounds r . A closed sublevel set is then a closed subset of $K\{a_r : 1 \leq r \leq R\}K$, which is compact.

If $c(v) = 0$, choose k_0 so that $w = \rho(k_0)v$ has no positive weights. For $r \geq 1$, orthogonality gives $\|\rho(a_r)w\|_B \leq \|w\|_B$. The matrices $a_r k_0$ escape compact sets as $r \rightarrow \infty$, so Φ_v is not proper.

Now suppose Φ_v is not proper and $B_T(v)$ is nonempty. Choose g_0 with $\|\rho(g_0)v\|_B < T$. The orbit map for $\rho(g_0)v$ is the composition of Φ_v with right translation, so it too is not proper. By the equivalence already proved, choose k_0 such that $w = \rho(k_0 g_0)v$ has only nonpositive weights. Its norm is still less than T . Continuity supplies $\delta > 0$ with

$$\sup_{|t| \leq \delta} \|\rho(n_t)\|_{\text{op}} \|w\|_B < T.$$

(If $w = 0$, any $\delta > 0$ works.) Therefore

$$\{k n_t a_r k_0 g_0 : k \in K, |t| \leq \delta, r \geq 1\} \subseteq B_T(v).$$

Right translation preserves Haar measure, and (3) shows that this full-dimensional set has measure $4\pi\delta \int_1^\infty dr/r = \infty$. $\square$

3 The root multiplicity criterion

Corollary 3. *Let f be a nonzero real binary form of degree $d \geq 1$. For any norm, all $B_T(f)$ have finite Haar volume if and only if every real projective root of f has multiplicity strictly less than $d/2$. If this condition fails, every nonempty Bombieri-norm ball has infinite volume.*

Proof. The equation $c(f) = 0$ means that, after some rotation, all coefficients with $j > d/2$ in (1) vanish. Equivalently, that rotated form is divisible by $y^{\lceil d/2 \rceil}$. This is precisely a root of multiplicity at least $\lceil d/2 \rceil$ at $[1 : 0]$. Conversely, a rotation moves any real projective root to $[1 : 0]$. Apply Theorem 2 and Lemma 1. Properness is independent of the choice of norm in finite dimension. $\square$

There is also a quantitative coercivity statement. If the maximum real-root multiplicity is $m < d/2$, taking $m = 0$ if there are no real roots, let Q_m project onto the monomials with $j \geq d - m$. For every $k \in K$, $Q_m \rho(k)f \neq 0$, since otherwise y^{m+1} divides the rotated form. Thus $c_m = \min_K \|Q_m \rho(k)f\|_B > 0$, and the proof of (4) gives

$$\|\rho(g)f\|_B \geq c_m \sigma(g)^{d-2m},$$

where $\sigma(g) \geq 1$ is the largest singular value of g .

Remark 4. The word *real* is essential. The positive definite quadratic $x^2 + y^2$ has a proper orbit map and compact stabilizer $\text{SO}_2(\mathbb{R})$, although its two complex projective roots have multiplicity $d/2$. The criterion is not a statement about complex geometric invariant theory.

4 A trivial stabilizer with infinite volume

Proposition 5. *For $f = x^2y$, the stabilizer in $\mathrm{SL}_2(\mathbb{R})$ is trivial and $m_G(B_T(f)) = \infty$ for every $T > 0$, in every norm. For the Bombieri norm and $0 < \epsilon < \sqrt{3}T$, the cutoff $r \geq \epsilon$ in (2) has volume*

$$m_G(B_T(f) \cap \{r \geq \epsilon\}) = 4\pi \left(\frac{\sqrt{T^2 - \epsilon^2/3}}{\epsilon} - \frac{1}{\sqrt{3}} \arccos \frac{\epsilon}{\sqrt{3}T} \right). \quad (5)$$

For $\epsilon \geq \sqrt{3}T$ the cutoff set is empty.

Proof. The two projective zero lines have multiplicities two and one, so a stabilizer element preserves each line and is $\mathrm{diag}(a, a^{-1})$. It multiplies x^2y by a , forcing $a = 1$. Direct substitution gives

$$\rho(n_t a_r) f = r x^2(t x + y), \quad \|\rho(k n_t a_r) f\|_B^2 = r^2(t^2 + 1/3). \quad (6)$$

For an arbitrary norm, the continuous function $\|\rho(k)(x^2(tx + y))\|$ is bounded by a positive constant C on $K \times [-1, 1]$. Hence the strip $k \in K$, $|t| \leq 1$, $0 < r < T/(2C)$ lies in $B_T(f)$ and has infinite measure by (3).

For the exact cutoff, (6) permits $|t| < \sqrt{T^2 - r^2/3}/r$ when $r < \sqrt{3}T$. Integrating first in θ, t gives

$$4\pi \int_{\epsilon}^{\sqrt{3}T} \frac{\sqrt{T^2 - r^2/3}}{r^2} dr.$$

Substitution $r = \sqrt{3}Tu$ evaluates the integral as (5). □

For fixed $T > 0$, the cutoff formula gives

$$m_G(B_T(f) \cap \{r \geq \epsilon\}) = \frac{4\pi T}{\epsilon} - \frac{2\pi^2}{\sqrt{3}} + O(\epsilon/T) \quad (\epsilon \downarrow 0).$$

The coordinate r is the Euclidean length of the first matrix column. This cutoff is not a canonical regularization and no quotient limit for the resulting measures is asserted.

Taking, for example, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the finite-stabilizer hypothesis preceding AIM Question 19 is satisfied, but normalized Haar measure on $B_T(f)$ does not exist for any $T > 0$. An escaping diagonal ray alone would not establish this fact, since that ray has Haar measure zero; the full-dimensional strip is decisive.

5 The critical boundary and scope

The strict inequality in Corollary 3 is necessary even for finite stabilizers. Consider $q = x^4 + x^2y^2 = x^2(x^2 + y^2)$. Its unique real projective root has multiplicity two. A real stabilizer element must preserve that root, hence has the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. The transformed quartic is

$$a^2(a^2 + b^2)x^4 + 2abx^3y + x^2y^2.$$

Equality with q forces $b = 0$ and $a^4 = 1$, so over $\mathbb{R}$ the stabilizer is exactly $\{I, -I\}$. In the coordinates (2),

$$\|\rho(g)q\|_B^2 = (r^4 + t^2)^2 + t^2 + \frac{1}{6}.$$

Its infimum is $1/6$ and is not attained. Thus $B_T(q)$ is empty for $T \leq 1/\sqrt{6}$, whereas it is nonempty and has infinite volume for $T > 1/\sqrt{6}$. Nonproperness therefore does not imply infinite volume at every positive radius for an arbitrary vector.

The results establish a complete normalization obstruction and a finite-volume criterion for the stated representations. They neither classify limits on G/Γ for proper orbit maps nor establish tightness, well-roundedness or mixing rates. AIM Question 19 under suitable corrected hypotheses is not declared solved.

The accompanying exact-arithmetic checker verifies the action convention, Haar Jacobians, polynomial stabilizers, norm identities, cutoff antiderivative and threshold controls. These tests supplement the proofs; they do not replace the quantified arguments.

References

- [1] American Institute of Mathematics, *Emerging applications of measure rigidity*, workshop problem list, version 1 November 2004, Section 4.3, Question 19 (N. Shah), p. 8. <https://aimath.org/WWN/measrigid/measrigid.pdf>.
- [2] `ulamai`, *UnsolvedMath*, version 1.6.0, record AIM-OTHER-0095, associated research report, Sections 7.4 and 9–10; frozen commit `c6e7f41b4eb6ce876feb83e7eb626be1b243fd7e`. <https://huggingface.co/datasets/ulamai/UnsolvedMath>.
- [3] A. Gorodnik and B. Weiss, *Distribution of lattice orbits on homogeneous varieties*, Geom. Funct. Anal. **17** (2007), 58–115. doi:10.1007/s00039-006-0583-6. Preprint: <https://arxiv.org/abs/math/0407345>.
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