

Sharp Tensor Nilpotence Bounds for Finite Crossed Products

Alper Ferudun*

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Abstract

Let a finite group G act on an additive braided rigid monoidal category $\mathcal{K}$ by additive strong monoidal autoequivalences. We prove that every tensor-nilpotent object of $\mathcal{K} \rtimes G$ has nilpotency index at most the order of the subgroup generated by its nonzero homogeneous degrees. The bound applies to arbitrary sums of homogeneous objects and requires no support theory or Noetherianity assumption. It is sharp for every finite group. In characteristic p , explicit permutation modules over $(C_p)^G$ give stable matrix units in a crossed product and realize the tensor-power pattern of every directed graph on G . A Hamilton path attains the bound $|G|$, while a four-group example shows that group exponent cannot replace group order. The corresponding statement for finite tensor categories bounds the first projective tensor power. This is a scoped tensor-projectivity result, not a solution of the general transfer problem from modular representations to finite tensor categories. The manuscript is unrefereed.

1 The order bound and its antecedents

Suppose that $\mathcal{K}$ is an additive braided rigid monoidal category with zero object and biadditive tensor product. Let G be a finite group acting coherently by additive strong monoidal autoequivalences T_g . The action need not preserve the chosen braiding. The crossed product $\mathcal{K} \rtimes G$ consists of finite direct sums of homogeneous objects, with multiplication

$$(A, g) \otimes (B, h) = (A \otimes T_g(B), gh). \quad (1)$$

We suppress the associativity and action coherence isomorphisms. An object X is *tensor nilpotent* if $X^{\otimes n} = 0$ for some $n \geq 1$. Its index $\nu(X)$ is the least such n ; set $\nu(0) = 1$ and $\nu(X) = \infty$ for nonnilpotent objects.

Theorem 1.1. *Write $X = \bigoplus_{g \in G} (X_g, g)$ and let $H = \langle g \in G : X_g \neq 0 \rangle$. If X is tensor nilpotent, then*

$$\nu(X) \leq |H| \leq |G|.$$

Equivalently, if $X^{\otimes |H|} \neq 0$, every positive tensor power of X is nonzero.

Here nilpotence concerns objects, not morphisms or central series of tensor categories. For a braided finite tensor category $\mathcal{C}$, the stable category $\underline{\mathcal{C}}$ has the structure needed in the theorem. An object is zero there exactly when it is projective. Thus the theorem gives the following projectivity test.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Corollary 1.2. *Let G act by exact monoidal autoequivalences on a braided finite tensor category $\mathcal{C}$. For $M = \bigoplus_g (M_g, g) \in \mathcal{C} \rtimes G$, let $H = \langle g : M_g \text{ is not projective} \rangle$. If some positive tensor power of M is projective, then $M^{\otimes |H|}$ is projective.*

For comparison, a braided finite tensor category has no nonprojective object with a projective tensor power: this is the duality-retract argument of Plavnik–Witherspoon [3, Theorem 5.2 and Remark 5.3] and Bergh–Plavnik–Witherspoon [4, Proposition 2.1]. The crossed product need not be braided. Benson–Witherspoon [2, Examples 3.2–3.3] already constructed projective-power examples, including every prescribed index using a cyclic acting group. We do not claim these existence results as new.

Huang–Vashaw [5, Section 5] develop crossed products of monoidal triangulated categories, identify the stable-category construction, classify two-sided ideals, and give spectral conditions for the existence of nilpotent objects. Miller [6, Section 8] treats one-sided ideals and permuted direct-sum examples. Our upper bound is an elementary quantitative consequence of a braided base and finite group prefixes; it requires none of the spectral finiteness hypotheses. Our sharpness construction uses actual finite-dimensional modules, with simple unit in the underlying abelian tensor category, and works for every finite acting group, including noncyclic groups.

The AIM fusion-category list asks how far modular representation theory of finite groups transfers to finite tensor categories [1, Problem 9.20]. The present result concerns only the crossed-product class above. It does not settle that broad question, nor assert an order bound for an arbitrary nonbraided base.

2 A repeated-prefix proof

The following familiar rigidity argument is included to make the precise categorical assumptions explicit.

Lemma 2.1. *In an additive braided rigid monoidal category with biadditive tensor product, $Z^{\otimes n} = 0$ for some $n \geq 1$ implies $Z = 0$. Consequently, if $A \otimes B \otimes C \neq 0$, then*

$$A \otimes B^{\otimes r} \otimes C \neq 0 \quad (r \geq 1). \quad (2)$$

Proof. Choose a dual Z^* with coevaluation into $Z \otimes Z^*$ and evaluation from $Z^* \otimes Z$. The triangle identity exhibits Z as a retract of $Z \otimes Z^* \otimes Z$. For $n \geq 2$, tensor this retraction with $Z^{\otimes(n-2)}$ and use braiding to reorder the target. Then $Z^{\otimes(n-1)}$ is a retract of $Z^{\otimes n} \otimes Z^*$. If $Z^{\otimes n} = 0$, the preceding power is therefore zero. Induction proves the first assertion. No categorical dimension is inverted.

If the expression in (2) were zero, braiding would give

$$(A \otimes B \otimes C)^{\otimes r} \cong (A \otimes B^{\otimes r} \otimes C) \otimes A^{\otimes(r-1)} \otimes C^{\otimes(r-1)} = 0.$$

The first assertion contradicts $A \otimes B \otimes C \neq 0$. □

Proof of Theorem 1.1. Put $h = |H|$ and suppose $X^{\otimes h} \neq 0$. Finite biadditivity expands this power into a direct sum of homogeneous words. Some word

$$W = (X_{g_1}, g_1) \otimes \cdots \otimes (X_{g_h}, g_h)$$

is nonzero. The $h + 1$ prefixes $s_0 = e$ and $s_i = g_1 \cdots g_i$ lie in the h -element group H , so $s_i = s_j$ for some $0 \leq i < j \leq h$. The intervening word has degree $g_{i+1} \cdots g_j = s_i^{-1} s_j = e$.

Write the prefix, middle and suffix as P , $B = (B_0, e)$ and $Q = (C_0, t)$, with $P = (A, s_i)$. Empty prefixes or suffixes mean the tensor unit. The underlying object of W is

$$A \otimes T_{s_i}(B_0) \otimes T_{s_i}(C_0) \neq 0.$$

Since B has degree e , the word $P \otimes B^{\otimes r} \otimes Q$ has underlying object

$$A \otimes T_{s_i}(B_0)^{\otimes r} \otimes T_{s_i}(C_0),$$

which is nonzero by Lemma 2.1. It is a direct summand of $X^{\otimes(h+(r-1)(j-i))}$. Hence nonzero powers exist at arbitrarily large exponents. A single zero power would force all subsequent powers to vanish, so X is not nilpotent. This proves the contrapositive, including $h = 1$. $\square$

Proof of Corollary 1.2. Projectivity in $\mathcal{C} \rtimes G$ is componentwise, since its underlying abelian category is a finite direct sum of copies of $\mathcal{C}$. Projectives form a tensor ideal, and each T_g preserves them. Consequently

$$\underline{\mathcal{C}} \rtimes G \simeq \underline{\mathcal{C}} \rtimes G$$

as monoidal categories, as in [5, Example 5.2]. Apply Theorem 1.1 after discarding the zero stable components. $\square$

3 Stable matrix units from permutation modules

Fix a finite group G of order $m \geq 2$, a prime p , and a field k of characteristic p . Let $P = (C_p)^G$, with coordinates indexed by $a \in G$. For each a , let U_a be the external tensor product of the trivial module in coordinate a and the regular kC_p -module in every other coordinate. Put $q = p^{m-1} = \dim_k U_a$.

Lemma 3.1. *Each U_a is nonprojective. For $n \geq 1$, the module $U_{a_1} \otimes \cdots \otimes U_{a_n}$ is nonprojective exactly when all a_i coincide. If all equal a , the product is isomorphic to $U_a^{\oplus q^{n-1}}$.*

Proof. The group algebra kP is local, so its finite projective modules are free. The dimension p^{m-1} is not divisible by $|P| = p^m$; thus U_a is nonprojective. Products can be computed coordinatewise. If at least two labels differ, every coordinate has a regular tensor factor. A regular group module tensored with any finite module is free: on $kC_p \otimes V$ the map $x \otimes v \mapsto x \otimes x^{-1}v$ identifies the diagonal action with the action on the first factor. The external product is therefore free over kP .

If all labels equal a , the restriction to the a -th cyclic factor is nonzero and trivial, so it is not projective. In every other coordinate the tensor product of n regular modules is p^{n-1} copies of the regular module. For example, on the permutation basis indexed by $(C_p)^n$, the change

$$(v_1, \dots, v_n) \mapsto (v_1, v_2 - v_1, \dots, v_n - v_1)$$

makes the diagonal translation act only on the first coordinate. Multiplying the multiplicities over the $m - 1$ regular coordinates gives q^{n-1} . $\square$

Let G act on P by left translation of its coordinate labels. Pullback along the inverse automorphism gives an exact symmetric monoidal action on $\text{Rep}_k(P)$ with $T_g U_a \cong U_{ga}$ and $T_g T_h \cong T_{gh}$. Its crossed product is the module category of the standard smash-coproduct Hopf algebra, of dimension mp^m ; the component formula is [2, Theorem 2.1]. This is a crossed product, not an equivariantization. There is no requirement that p be coprime to $|G|$.

In $\underline{\text{Rep}}_k(P) \rtimes G$, define

$$E_{ab} = (U_a, ab^{-1}) \quad (a, b \in G). \quad (3)$$

Proposition 3.2. *The nonzero stable objects E_{ab} satisfy*

$$E_{ab} \otimes E_{cd} \cong \begin{cases} 0, & b \neq c, \\ E_{ad}^{\oplus q}, & b = c. \end{cases} \quad (4)$$

For any matrix $A = (A_{ab})_{a,b \in G}$ with nonnegative integer entries, let $M_A = \bigoplus_{a,b} E_{ab}^{\oplus A_{ab}}$. For every $n \geq 1$ there is a stable isomorphism

$$M_A^{\otimes n} \cong \bigoplus_{a,b \in G} E_{ab}^{\oplus q^{n-1}(A^n)_{ab}}. \quad (5)$$

In particular, $M_A^{\otimes n} = 0$ stably if and only if $A^n = 0$ as an integer matrix.

Proof. The underlying module of the left side of (4) is $U_a \otimes U_{ab^{-1}c}$. Its two labels coincide exactly when $b = c$, even when G is nonabelian. Lemma 3.1 gives zero stably otherwise. If $b = c$, the product is $U_a^{\oplus q}$ in degree $(ab^{-1})(bd^{-1}) = ad^{-1}$, proving (4). Finite distributivity and induction now give (5). Each displayed multiplicity is an integer number of direct-sum copies, not multiplication by a field scalar. In particular it does not vanish in characteristic p . Since each E_{ab} is nonzero and a summand of a zero object is zero, no cancellation is possible. $\square$

The matrix-unit terminology describes (4); it does not assert a unital equivalence with a matrix category. Neither division by q nor a claim of a new matrix-nilpotence identity is involved.

4 Directed graphs and sharpness for every group

Take A to be the adjacency matrix of a directed multigraph Γ on G , with A_{ab} edges from a to b . Loops are allowed. Equation (5) realizes the ordinary directed-walk counts as stable tensor multiplicities. Denote the corresponding object by M_Γ and use the same notation for its specified module representative in $\text{Rep}_k(P) \rtimes G$.

Theorem 4.1. *The tensor power $M_\Gamma^{\otimes n}$ is nonprojective exactly when Γ has a directed walk of length n . If Γ is acyclic with longest path of length L , its tensor-projectivity index is $L + 1$, with index one for the graph with no edges. If Γ has a directed cycle, no positive tensor power is projective. In particular, for every finite G of order $m \geq 2$, an object of tensor-projectivity index exactly m exists.*

Proof. For a nonnegative adjacency matrix, $(A^n)_{ab}$ counts length- n directed walks from a to b . Proposition 3.2 proves the first assertion. In a finite acyclic graph no vertex can repeat in a directed walk, so walks are paths. The last nonzero power is L . A cycle permits walks of every length. Finally, a directed Hamilton path through the m vertices has longest path $m - 1$, giving index m . $\square$

The Hamilton-path example also has degree-generated subgroup G : if its successive vertices are $a_1, \dots, a_m$, its degrees generate a subgroup containing all products $a_1 a_j^{-1}$, which range over G . Thus it attains the sharper bound in Theorem 1.1.

For a concrete distinction between order and exponent, write the four-group additively as $G = \{0, a, b, a + b\}$ and take the path

$$0 \longrightarrow a \longrightarrow a + b \longrightarrow b.$$

The object has degree- a component $U_0 \oplus U_{a+b}$ and degree- b component U_a . In characteristic two its dimension is $3 \cdot 2^3 = 24$. Its cube is nonprojective and its fourth power is projective, although $\exp(G) = 2$.

Nilpotent objects need not be closed under direct sums. For $G = C_2$, each of the single edges $e \rightarrow g$ and $g \rightarrow e$ gives a square-zero stable object. Their sum is the directed two-cycle and is not nilpotent. This explains why a bound for one homogeneous summand alone does not prove Theorem 1.1.

If the action is trivial, no nonzero tensor-nilpotent object exists in $\mathcal{K} \rtimes G$: a nonzero homogeneous summand has nonzero powers by Lemma 2.1. On the other hand, dropping braidedness of the base invalidates a bound solely in terms of the new acting group: take a known nonzero nilpotent example as base and let the new acting group be trivial.

5 Exact checks and scope

The accompanying standard-library Python program performs 136,296 explicit exact-arithmetic checks. It exhausts all 66,066 directed graphs with loops on one to four labelled vertices, comparing Boolean powers with a separate topological-sort longest-path computation. It also compares direct twisted word expansion with path counts for $C_2, C_3, C_4, C_2 \times C_2, S_3$, exhaustively for edge subsets on C_2, C_3 and on recorded deterministic samples otherwise. Small permutation-module norm computations over $\mathbb{F}_2$ and $\mathbb{F}_3$ test the projectivity criterion. Normal and optimized Python give identical outputs. These are finite regression tests, not a replacement for the proofs or formal verification.

The categorical constructions, braided retract, cyclic examples and elementary matrix path rule have the antecedents cited above. The contribution claimed here is the arbitrary-object order bound and its sharp realization by these finite modules for each finite acting group. No absolute-priority claim is made. The broad AIM transfer problem remains open beyond this specified result.

The author used AI tools in developing the arguments, searching literature, preparing the exposition and implementing the checks, and takes responsibility for the manuscript. The work has not undergone independent peer review or proof-assistant verification.

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