

Finite Bipartite Graphs as Induced Subgraphs of Subfactor Principal Graphs

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Abstract

Every finite bipartite simple graph occurs as a vertex-induced, unrooted subgraph of the principal graph of an irreducible finite-depth inclusion of hyperfinite type II_1 factors. For a graph with m left vertices and n right vertices, a direct-product finite-group construction has index $q3^n$ and depth at most four, where $q \leq m + 1$ is an explicit neighborhood-multiplicity parameter. The selected vertices have depths two and three, respectively. We compute all depth ranks and the complete adjacency spectrum of the ambient principal graph. A separate amplification realizes every finite bipartite multigraph as an ordinary subgraph, with edge deletion allowed. The ingredients are classical finite-group subfactor theory and elementary representation theory. The conclusions concern finite unrooted subgraphs across variable indices, not fixed-index rooted extensions or the full AIM polymer programme. This preprint is unrefereed and makes no absolute priority claim.

1 Scope and construction

The AIM fusion-category workshop asks for a structural theory of subfactor principal graphs and of the graphs they contain [1, Problem 9.31]. Here we distinguish a selected unrooted subgraph from a complete rooted truncation. Deleting vertices need not preserve the root, all vertices through a given depth, statistical dimensions, duality, or the companion dual principal graph. Those data are central to rooted extension and obstruction methods [5, Section 2.3].

We allow isolated vertices, empty parts, and disconnected graphs. A vertex-induced embedding preserves both the presence and the absence of edges between every pair of selected vertices. A graph is nonempty when it has at least one vertex.

Theorem 1. *Let $B = (L, R, E)$ be a nonempty finite bipartite simple graph, with $|L| = m$ and $|R| = n$. Write*

$$a_S = \#\{i \in L : N_B(i) = S\}, \quad q = \max \left(1, a_\emptyset + 1, \max_{\emptyset \neq S \subseteq R} a_S \right),$$

where a maximum over an empty collection is zero. There is an irreducible finite-depth inclusion of hyperfinite type II_1 factors whose principal graph contains B as a vertex-induced subgraph. Its index is $q3^n$ and its depth is at most four. The selected left vertices have depth two and the selected right vertices have depth three.

Since $q \leq m + 1$, this gives an index bound $(m + 1)3^n$. Interchanging the two parts gives a second construction and bound. Neither bound is asserted to be optimal. The graph with no vertices embeds by selecting no vertices in any proper finite-group example.

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We use the classical finite-group fixed-point construction [2, p. 6][3, p. 13]. If an outer finite group G acts on the hyperfinite factor $\mathcal{R}$, and $H \leq G$, then

$$N = \mathcal{R}^G \subset M = \mathcal{R}^H, \quad [M : N] = [G : H]$$

is irreducible and has finite depth. Its principal graph is the component of the restriction graph containing 1_G : even vertices are in $\text{Irr}(G)$, odd vertices are in $\text{Irr}(H)$, and edge multiplicity is $[\text{Res}_H^G \rho : \eta]$. This is the fixed-point orientation; we do not interchange it with a crossed-product dual graph.

For completeness, an outer action exists without an additional realization hypothesis. Take the infinite tensor product of $M_{|G|}(\mathbb{C})$ with normalized trace, and let G act in every factor by the adjoint left-regular representation. For $g \neq 1$, the projection onto the identity group basis vector, placed in successive remote factors, is a bounded central sequence. Its image is an orthogonal projection, so its displacement has a fixed positive 2-norm. An inner automorphism moves any bounded central sequence by $o(1)$ in 2-norm. Thus every nonidentity element acts outerly. Fixed-point subfactor theory then supplies the properties just stated. These are classical ingredients of our construction.

2 An exact restriction matrix

Put

$$G = S_3^n \times C_q, \quad H = C_2^n \times \{1\},$$

where each C_2 is generated by a transposition in its S_3 factor, and C_1 denotes the trivial group. Write T, S, W for the trivial, sign, and two-dimensional standard complex representations of S_3 , and η for the nontrivial character of C_2 . Then

$$T|_{C_2} = 1, \quad S|_{C_2} = \eta, \quad W|_{C_2} = 1 \oplus \eta. \quad (1)$$

Indeed, W is the sum-zero subspace of the permutation representation on $\mathbb{C}^3$, and a transposition has eigenvalues $1, -1$ there. A three-cycle has distinct eigenvalues ζ_3, ζ_3^{-1} and a transposition exchanges the two eigenspaces, proving irreducibility. The three irreducibles exhaust S_3 , since $1^2 + 1^2 + 2^2 = 6$.

Let $\lambda_c, c \in \mathbb{Z}/q\mathbb{Z}$, be the cyclic characters. The direct-product irreducibles are

$$\rho_{\mathbf{u},c} = u_1 \boxtimes \cdots \boxtimes u_n \boxtimes \lambda_c, \quad \mathbf{u} \in \{T, S, W\}^n.$$

The subgroup irreducibles are $\eta_\alpha = \boxtimes_{j=1}^n \eta^{\alpha_j}$ for $\alpha \in \{0, 1\}^n$. All restriction multiplicities are zero or one. Equation (1) gives the exact rule

$$[\rho_{\mathbf{u},c}|_H : \eta_\alpha] = 1 \iff \begin{cases} \alpha_j = 0 & \text{if } u_j = T, \\ \alpha_j = 1 & \text{if } u_j = S, \end{cases} \text{ for all } j. \quad (2)$$

An entry $u_j = W$ imposes no condition. The cyclic label is invisible on H , and therefore duplicates a branching row without identifying its even vertex.

Proof of Theorem 1. Number the right vertices by $1, \dots, n$. For $i \in L$, choose the word $\mathbf{u}_i$ with W in coordinates of $S_i = N_B(i)$ and T elsewhere. For each fixed nonempty neighborhood, assign distinct cyclic labels to its rows. For the empty neighborhood, assign distinct *nonzero* labels. The definition of q guarantees both assignments; labels may be reused for different neighborhoods.

Select the even vertices $\rho_i = \rho_{\mathbf{u}_i, c_i}$ and the odd vertices η_{e_j} , where e_j is the j th binary unit vector. The even vertices are pairwise inequivalent: different neighborhoods give different words, and equal

neighborhoods have different labels. They are nontrivial, because either the word contains W , or its cyclic label is nonzero. The selected odd vertices are distinct and nontrivial. By (2),

$$[\rho_i|_H : \eta_{e_j}] = \begin{cases} 1 & j \in N_B(i), \\ 0 & j \notin N_B(i). \end{cases}$$

This proves the required induced incidence matrix, including its zero entries.

The root $1_G = \rho_{(T, \dots, T), 0}$ is adjacent only to 1_H . Every selected even vertex contains 1_H on restriction, and is not the root, so has depth two. If $n > 0$, the even vertex $\rho_{(W, \dots, W), 0}$ restricts to the regular representation of H . It connects 1_H to every odd vertex, so all nontrivial odd vertices have depth three. Every even vertex has a nonzero restriction constituent and hence lies at depth at most four. Thus the entire restriction graph is connected and is the principal graph of the fixed-point inclusion.

Finally, $[G : H] = q3^n$. If $n > 0$, this is greater than one. If $n = 0$ and $m > 0$, then $q = m + 1 \geq 2$; the restriction graph is a star with q even vertices and one odd vertex. This also realizes the selected isolated left vertices at depth two. The cases $m = 0$ and isolated right vertices are already covered by the connectedness argument. The inclusion is therefore proper for every nonempty input. $\square$

Corollary 2. *Across all finite indices, the finite simple unrooted induced subgraphs of irreducible finite-depth hyperfinite subfactor principal graphs are exactly the finite bipartite simple graphs.*

Proof. Principal graphs are bipartite by their even and odd bimodule classes. The converse is Theorem 1. $\square$

3 Depth ranks and spectrum

Proposition 3. *The full principal graph in Theorem 1 has depth ranks*

$$r_0 = 1, \quad r_1 = 1, \quad r_2 = q2^n - 1, \quad r_3 = 2^n - 1, \quad r_4 = q(3^n - 2^n).$$

Its nonzero adjacency eigenvalues are $\pm \sqrt{q3^{n-k}}$, for $0 \leq k \leq n$, with multiplicity $\binom{n}{k}$ for each sign. Zero has multiplicity $q3^n - 2^n$. In particular its squared adjacency norm is the index $q3^n$.

Proof. A row contains 1_H exactly when its word has no S . There are $q2^n$ such rows, including the root. Every other even row appears at depth four; every nontrivial odd vertex appears at depth three, as proved above. This gives the ranks, including $n = 0$, when trailing zero ranks are omitted.

Let C be the even-by-odd restriction matrix. For binary words α, β , an agreeing coordinate admits two symbols in a row containing both characters, whereas a disagreeing coordinate admits only W . Consequently

$$(C^\top C)_{\alpha, \beta} = q 2^{\#\{j: \alpha_j = \beta_j\}}, \quad C^\top C = q \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{\otimes n}.$$

The two eigenvalues of the 2 by 2 factor are 3 and 1. The Gram eigenvalues are therefore $q3^{n-k}$ with multiplicity $\binom{n}{k}$. They are positive, so C has rank 2^n . The adjacency matrix $\begin{pmatrix} 0 & C \\ C^\top & 0 \end{pmatrix}$ has the positive and negative singular values of C and $q3^n - 2^n$ further zero eigenvalues. For $n = 0$, the tensor power is the one-dimensional identity and this is the usual spectrum of a star. $\square$

For example, a single edge uses (S_3, C_2) and the principal graph is the five-vertex path, with the selected edge between depths two and three. A graph consisting of m isolated left vertices uses $(C_{m+1}, \{1\})$. These examples also explain why the nonzero-label requirement for empty neighborhoods is necessary: otherwise an isolated selected vertex could coincide with the root.

4 Ordinary bipartite multigraphs

The induced assertion above concerns simple graphs. Edge amplification gives a separate, weaker assertion for multigraphs. Finite-group Frobenius examples of the kind used below are classical [4, Section 3 and Figure 2].

Proposition 4. *Every finite bipartite multigraph is an ordinary subgraph of the principal graph of an irreducible finite-depth hyperfinite subfactor. If both parts have positive sizes m, n and the maximum edge multiplicity is $b \geq 1$, an index $m3^n 9^r$ suffices for any $r \geq 0$ with $2^r \geq b$.*

Proof. First realize $K_{m,n}$ by Theorem 1. All its left neighborhoods agree and are nonempty, so $q = m$. The selected restriction multiplicities are all one.

Let $G_0 = \mathbb{F}_3^2 \rtimes Q_8$ and $H_0 = Q_8$. Over $\mathbb{F}_3$, the matrices

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

satisfy $I^2 = J^2 = -1$ and $IJ = -JI$. They generate Q_8 ; every nonidentity element fixes no nonzero vector. Hence the eight nontrivial characters of $\mathbb{F}_3^2$ form a free orbit. Induce one character to G_0 , obtaining ρ_0 of dimension eight. Its distinct normal-subgroup weights, permuted transitively, prove irreducibility: an invariant subspace is a sum of weight spaces and must contain either all or none. Restriction to H_0 is its regular representation, since H_0 acts regularly on the cosets of the normal subgroup. In particular, the two-dimensional irreducible σ of Q_8 occurs twice.

Take the direct product of the original group pair with r copies of (G_0, H_0) . Select the even vertices $\rho_i \boxtimes \rho_0^{\boxtimes r}$ and odd vertices $\eta_{e_j} \boxtimes \sigma^{\boxtimes r}$. Restriction multiplicities multiply, so every selected pair has multiplicity 2^r . The vertices remain pairwise distinct. Every selected even restriction contains the trivial subgroup representation, putting that vertex at depth two; every selected odd vertex is nontrivial and adjacent to these vertices. Thus all are in the root component and hence the principal graph.

Delete unwanted edges and excess parallel edges to obtain the prescribed multigraph. Indices multiply to $m3^n 9^r$. Edgeless inputs and inputs with an empty part are covered by the simple-graph construction. This proof allows edge deletion; it does not assert induced multigraph universality. $\square$

5 Relation to rooted and bounded index questions

Our index varies with the graph, and the selected subgraph excludes the distinguished root. No rooted forbidden-pattern statement or bounded-index obstruction is contradicted. The workshop's polymer terminology and the later unspecified bad-seed diagram are not defined by the finite-subgraph result. We therefore do not claim to solve the full AIM source question.

For comparison, if the index is at most D , the adjacency norm is at most $\sqrt{D}$ [3, p. 15]. For a vertex v with edge multiplicities $m(v, w)$, this implies

$$\sum_w m(v, w)^2 \leq D, \quad \sum_w m(v, w) \leq \lfloor D \rfloor, \quad m(v, w) \leq \lfloor \sqrt{D} \rfloor.$$

The first inequality follows by applying adjacency to the unit vector at v ; the others use nonnegative integral multiplicities. Also, the number of vertices at exact depth k is at most $\lfloor D^{k/2} \rfloor$: the alternating k -fold basic bimodule has dimension at most $D^{k/2}$, every depth- k vertex occurs with positive integral multiplicity, and each simple statistical dimension is at least one. These elementary necessary bounds permit finite enumeration of complete rooted balls at any fixed radius, but do not decide flatness

or extension to an actual standard invariant. The established odometer framework addresses rooted extension data [5]; it is a different question from selecting an unrooted induced subgraph.

The present contribution is an explicit elementary realization with an index parameter, depth bound, and spectral calculation. The finite-group construction, tensor-product representation theory, and Frobenius amplification are not new ingredients. A bounded primary-source comparison did not establish historical priority for their combination here; the argument may be familiar to specialists. No exhaustive novelty clearance is claimed.

The accompanying source archive contains exact-arithmetic diagnostics for all 689 labeled bipartite simple graphs with part sizes from zero through three, additional repeated-neighborhood cases, and full branching and spectral checks. The final direct-product checker executed 441,362 assertions under each of Python 3.13.5 and 3.9.6, with identical mathematical reports. These are finite diagnostics, not a substitute for the all-input proof above or independent peer review. Earlier constructions and their checks are retained for reproducibility.

AI assistance was used for mathematical exploration, literature comparison, drafting, and exact-checker development. The result was self-audited within this project; it has not been independently peer reviewed or formally verified. This is an unrefereed preprint released under CC BY 4.0.

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