

# Brauer–Picard Groups and Module Orbits for Twisted Cyclic Categories

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## Abstract

For every cyclic group  $C_n$  and every associator class, we give an explicit description of the Brauer–Picard group of  $\text{Vec}_{C_n}^{\omega_t}$  and its action on indecomposable module categories. The module labels are the divisors of  $g = \gcd(n, t)$ . For  $p^k \parallel n$  and  $s = \min(v_p(t), k)$ , the local group is the unit subgroup  $U = \{a : a^2 \equiv 1 \pmod{p^{k-s}}\}$  if  $s = 0$ , and the inversion semidirect product  $U \rtimes C_2$  if  $s > 0$ . The units fix every module label and the involution sends its local exponent  $j$  to  $s - j$ . In particular, there are  $\prod_{p|n} (\lfloor s/2 \rfloor + 1)$  orbits, and the action is transitive exactly when  $g$  is squarefree. The proof computes the center as a metric group, classifies all its Lagrangian subgroups, including the noncyclic ones, and constructs the full orthogonal group. All formulas include the 2-primary cases. The categorical machinery, untwisted specializations, and earlier cyclic-Lagrangian counts are credited explicitly.

## 1 Statement and scope

Let  $C_n = \mathbb{Z}/n\mathbb{Z}$ , with standard representatives  $0, \dots, n-1$ . For  $n \geq 2$  and  $0 \leq t < n$ , fix the normalized cocycle

$$\omega_t(a, b, c) = \exp\left(\frac{2\pi i t a}{n} \left\lfloor \frac{b+c}{n} \right\rfloor\right), \quad \mathcal{C}(n, t) = \text{Vec}_{C_n}^{\omega_t}. \quad (1)$$

These representatives exhaust  $H^3(C_n, \mathbb{C}^\times)$ . All categories and module categories below are finite semisimple over  $\mathbb{C}$ ; module categories are classified up to module equivalence. The Brauer–Picard group consists of equivalence classes of invertible bimodule categories, acting by relative tensor product.

Set  $g = \gcd(n, t)$ . For each prime power  $p^k \parallel n$ , put

$$s_p = \min(v_p(t), k), \quad U_{p,k,s_p} = \{a \in (\mathbb{Z}/p^k\mathbb{Z})^\times : a^2 \equiv 1 \pmod{p^{k-s_p}}\}. \quad (2)$$

Here  $v_p(0) = \infty$ , so  $s_p = k$  when  $t = 0$ . A congruence modulo 1 is vacuous. Write  $\omega(g)$  for the number of distinct prime divisors of  $g$ , with  $\omega(1) = 0$ .

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**Theorem 1.1.** *The following formulas hold for every  $n, t$  as above.*

1. *The indecomposable module categories are  $M_d$ , indexed by  $d \mid g$ , and  $M_d$  has  $n/d$  simple objects.*
2. *There is an isomorphism*

$$\mathrm{BrPic}(\mathcal{C}(n, t)) \cong \prod_{p^k \parallel n} B_{p, k, s_p}, \quad B_{p, k, s} = \begin{cases} U_{p, k, 0}, & s = 0, \\ U_{p, k, s} \rtimes_{a \mapsto a^{-1}} C_2, & s > 0. \end{cases} \quad (3)$$

3. *In this presentation, each unit subgroup fixes every  $M_d$ . The local  $C_2$  factor replaces  $v_p(d) = j$  by  $s_p - j$ , leaving all other prime exponents unchanged. Consequently*

$$\#(\mathrm{BrPic}(\mathcal{C}(n, t)) \setminus \{M_d\}) = \prod_{p \mid n} \left( \left\lfloor \frac{s_p}{2} \right\rfloor + 1 \right), \quad (4)$$

$$|\mathrm{BrPic}(\mathcal{C}(n, t))| = 2^{\omega(g)} \prod_{p^k \parallel n} |U_{p, k, s_p}|, \quad (5)$$

$$|\mathrm{Stab}(M_d)| = 2^{\#\{p \mid n : s_p > 0, \ 2v_p(d) = s_p\}} \prod_{p^k \parallel n} |U_{p, k, s_p}|. \quad (6)$$

*The action is transitive if and only if  $g$  is squarefree.*

For  $n = 1$  there is one module category and the group is trivial. Theorem 1.1 is a classification for this cyclic family, not a classification for all pointed or all fusion categories. The AIM request to classify module categories and Brauer–Picard groups for known examples [1, Examples, Problem 7.1] is broader. It appears as **AIM-OTHER-0057** in the frozen **ulamai/UnsolvedMath** v1.6.0 corpus; that general request remains open beyond the family considered here.

## Relation to established results

The subgroup classification of module categories is due to Ostrik [2, Example 2.1]. Etingof–Nikshych–Ostrik [3, Theorem 1.1] identify the Brauer–Picard group with braided autoequivalences of the center; the untwisted abelian orthogonal-group description is already their Corollary 1.2. Pointed braided categories are classified by quadratic groups [4, Theorem 8.4.12]. These are inputs, not new theorems here.

Mason–Ng [5] study the metric groups of twisted doubles and metabolizer orbits, including a homogeneous odd-primary result in their Theorem 13.7. Nikshych–Riepel [6] develop the categorical Lagrangian Grassmannian method. The cyclic cocycles and projective characters used below also occur in [7, Equations (47), (66)–(68)]. Meisma–Shinder [8, Proposition 3.6 and Lemma 3.8] count the cyclic Lagrangian subgroups of the same quadratic groups in lattice notation. Their definition in that discussion requires a Lagrangian to be cyclic; ours does not. Nikshych [9, Section 5.3, Theorem 5.14 and Corollary 5.15] gives the untwisted homocyclic orbit classification, with earlier attribution. In particular, the  $t = 0$  specialization of (4) is known.

The calculation here combines explicit orthogonal-group generators and the action on every Lagrangian for arbitrary cyclic associator, including nonzero twists and  $p = 2$ . No absolute-priority claim is made.

## 2 Module categories and the center

**Lemma 2.1.** *The module labels and ranks are as in Theorem 1.1(1).*

*Proof.* The subgroup  $H_d = \langle n/d \rangle$  of  $C_n$  has order  $d$ . Substituting multiples of  $n/d$  into (1) shows that its restriction is the standard cocycle of parameter  $t$  on  $C_d$ . It is pointwise 1 if  $d \mid t$ . Conversely,

$$\prod_{j=0}^{d-1} \omega_t(n/d, (n/d)j, n/d) = \exp(2\pi it/d).$$

For a normalized 3-coboundary on  $C_d$ , the corresponding product telescopes to 1. Thus triviality of the restricted cohomology class requires  $d \mid t$ .

Ostrik's classification uses a subgroup with trivial restricted associator and a torsor over  $H^2$  of that subgroup. Here  $H^2(C_d, \mathbb{C}^\times) = 0$ : a central extension splits after rescaling a lift of a generator by a  $d$ -th root of its  $d$ -th power. There are no further conjugacy identifications. The simple objects are indexed by  $C_n/H_d$ , proving the claim.  $\square$

We use additive quadratic forms with values in  $\mathbb{Q}/\mathbb{Z}$ , and write  $B(x, y) = Q(x+y) - Q(x) - Q(y)$  for the polar pairing.

**Proposition 2.2.** *The center of  $\mathcal{C}(n, t)$  is pointed, with metric group*

$$A(n, t) = \langle e, f : ne = 0, nf = 2te \rangle, \quad Q(af + be) = \frac{ta^2}{n^2} + \frac{ab}{n} \pmod{\mathbb{Z}}. \quad (7)$$

*Its order is  $n^2$ , and  $\text{BrPic}(\mathcal{C}(n, t)) \cong O(A(n, t), Q)$ .*

*Proof.* For flux  $a$ , all projective characters are

$$\rho_{a,b}(x) = \exp\left(2\pi i \left(\frac{tax}{n^2} + \frac{bx}{n}\right)\right), \quad 0 \leq a, b, x < n. \quad (8)$$

Indeed, the multiplier is  $\omega_t(a, x, y)\omega_t(x, y, a)/\omega_t(x, a, y) = \omega_t(a, x, y)$ ; direct division of the characters at  $x, y$  by the character at  $x + y \pmod{n}$  gives it. Untwisting by  $\rho_{a,0}$  reduces the representations to ordinary characters of  $C_n$ , so these  $n^2$  characters exhaust the simples.

The associator contribution to the tensor product of fluxes  $a, c$  at  $x$  is

$$\gamma_x(a, c) = \frac{\omega_t(a, c, x)\omega_t(x, a, c)}{\omega_t(a, x, c)} = \exp\left(\frac{2\pi itx}{n} \left\lfloor \frac{a+c}{n} \right\rfloor\right).$$

Multiplying (8) for  $a, b$  and  $c, d$ , including this contribution, gives

$$(a, b) + (c, d) = \left(a + c \pmod{n}, b + d + 2t \left\lfloor \frac{a+c}{n} \right\rfloor \pmod{n}\right). \quad (9)$$

Thus  $e = (0, 1)$  and  $f = (1, 0)$  satisfy the relations in (7). The carry is  $2t$ , with one contribution from each of the character normalization and the tensor associator. Self-braiding is  $\rho_{a,b}(a)$ , giving  $Q$ .

The form is well defined under both relations, and

$$B(e, e) = 0, \quad B(e, f) = 1/n, \quad B(f, f) = 2t/n^2.$$

An element pairing trivially with  $e$  has flux 0 modulo  $n$ ; pairing it with  $f$  then forces its charge to be 0. The form is therefore nondegenerate. The center theorem [3, Theorem 1.1] and the equivalence between pointed braided categories and premetric groups [4, Theorem 8.4.12] identify the full automorphism group with  $O(A, Q)$ . In particular, this is not just a map onto a subgroup: two lifts of the same isometry differ by a symmetric 2-cocycle, which is a coboundary.  $\square$

### 3 All Lagrangian subgroups and the module labels

A subgroup of  $A(n, t)$  is called Lagrangian if it is  $Q$ -isotropic and has order  $n$ . No cyclicity assumption is imposed.

**Proposition 3.1.** *All Lagrangian subgroups are*

$$L_d = \left\langle de, \frac{n}{d}f - \frac{t}{d}e \right\rangle, \quad d \mid g. \quad (10)$$

If  $c_d = \gcd(d, n/d, t/d)$ , then

$$L_d \cong C_{c_d} \times C_{n/c_d}. \quad (11)$$

The full-center correspondence sends  $M_d$  to the Lagrangian algebra supported on  $L_d$ , and identifies the Brauer–Picard action with the orthogonal action on these subgroups.

*Proof.* Project  $A(n, t)$  onto  $C_n$  by the  $f$  coordinate. If a Lagrangian  $L$  has image of order  $d$ , its kernel has order  $n/d$  and equals  $\langle de \rangle$ . A lift of the standard generator of the image has the form  $h = (n/d)f + be$ . Isotropy gives

$$Q(h) = t/d^2 + b/d = 0 \pmod{\mathbb{Z}},$$

which is equivalent to  $d \mid t$  and  $b \equiv -t/d \pmod{d}$ . This proves uniqueness and the necessity of (10).

Conversely its generators are isotropic and pair integrally. With  $v = de$  and  $h = (n/d)f - (t/d)e$ , the relations are

$$(n/d)v = 0, \quad dh = (t/d)v.$$

The projection and kernel show that the order is  $n$ . The relation matrix has determinant of absolute value  $n$  and entry  $\gcd c_d$ ; Smith normal form gives (11).

For the label identification,  $\omega_t|_{H_d} = 1$ , so choose the ordinary algebra on  $H_d$ . A central simple  $(a, b)$  maps nontrivially into its full center precisely when  $a \in H_d$  and its half-braiding with every  $h \in H_d$  is 1. Write  $a = (n/d)k$ . Evaluating (8) at  $n/d$  gives  $b \equiv -tk/d \pmod{d}$ , exactly the elements of  $L_d$ . There are  $n$  such simples.

An isotropic subgroup has a unique commutative algebra structure up to isomorphism. After trivializing the restricted braided category, two structures differ by a symmetric 2-cocycle, hence a coboundary. Thus no algebra decorations are omitted. The full-center correspondence and its equivariance [4, Section 8.27.2] give the asserted action.  $\square$

## 4 The complete orthogonal group

### 4.1 Reduction to a prime power

Fix  $N = p^k \parallel n$  and write  $n = Nm$ . The exponent of  $A(n, t)$  divides  $n^2$ . Choose  $\epsilon \equiv 1 \pmod{N^2}$  and  $\epsilon \equiv 0 \pmod{m^2}$ . Multiplication by  $\epsilon$  is the projection onto the  $p$ -primary subgroup. There set

$$E = \epsilon e, \quad F = m\epsilon f.$$

These elements generate the component, since  $m$  is invertible on a  $p$ -group, and satisfy

$$NE = 0, \quad NF = 2tE, \quad Q(E) = 0, \quad Q(F) = t/N^2, \quad B(E, F) = 1/N.$$

The last equalities are modulo  $\mathbb{Z}$ , using  $\epsilon^2 \equiv 1 \pmod{N^2}$ . If  $t = t_p + Nh$  with  $0 \leq t_p < N$ , replacing  $F$  by  $F - hE$  gives  $A(N, t_p)$ . Its order is  $N^2$ , so the entire component has been identified.

Distinct primary subgroups are orthogonal: the order of a pairing value divides the orders of both elements. Every primary subgroup is characteristic. Consequently

$$O(A(n, t), Q) = \prod_{p^k \parallel n} O(A(p^k, t_p), Q). \quad (12)$$

The local projection order of  $L_d$  is  $p^{v_p(d)}$ , so uniqueness in Proposition 3.1 identifies its local label. It remains to compute the prime-power group and its action.

## 4.2 The stabilizer of the canonical cyclic subgroup

Suppose now  $n = p^k$ ,  $s = \min(v_p(t), k)$ , and write  $U = U_{p,k,s}$ . An isometry stabilizing  $\langle e \rangle$  has

$$e \mapsto ve, \quad f \mapsto af + be$$

for units  $a, v$  modulo  $n$ . Preservation of  $B(e, f)$  implies  $av \equiv 1 \pmod{n}$ , and preservation of  $Q(f)$  implies

$$t(a^2 - 1) + nab \equiv 0 \pmod{n^2}. \quad (13)$$

Thus  $a \in U$  and  $b$  is uniquely determined modulo  $n$  by

$$b_a \equiv a^{-1} \frac{t(1 - a^2)}{n} \pmod{n}. \quad (14)$$

Take standard representatives for  $a$  in this formula. A change of representative changes the charge coordinate consistently with the relation  $nf = 2te$ .

Conversely, define  $S_a(e) = a^{-1}e$  and  $S_a(f) = af + b_a e$ . They preserve  $Q$  and  $B$  on the generators. They preserve  $nf = 2te$  because (13) implies  $n \mid t(a - a^{-1})$ . They are invertible on  $\langle e \rangle$  and on the quotient, hence on  $A$ . Uniqueness of the lift gives  $S_a S_c = S_{ac}$ . We have proved

$$\text{Stab}_{O(A, Q)}(\langle e \rangle) \cong U. \quad (15)$$

Every  $S_a$  fixes every  $L_{p^j}$ : multiplication by a unit preserves its projection order, which uniquely determines the Lagrangian by Proposition 3.1.

## 4.3 An involution and exhaustiveness

First assume  $0 < s < k$ , write  $t = p^s u$ , and choose  $r$  with  $ru \equiv 1 \pmod{n}$ . Define

$$R(f) = f, \quad R(e) = -e + rp^{k-s}f. \quad (16)$$

The map preserves the relations of  $A$  and satisfies  $R^2 = 1$ . For instance  $nR(e) = 2ru ne = 0$ , and the other relation reduces to  $4t(ru - 1)e = 0$ . Its quadratic and polar values are

$$Q(R(e)) = \frac{r(ru - 1)}{p^s} = 0, \quad B(R(f), R(e)) = \frac{2ru - 1}{n} = \frac{1}{n}, \quad Q(R(f)) = Q(f) \pmod{\mathbb{Z}}.$$

Thus  $R$  is an isometry, including for  $p = 2$ .

For  $0 \leq j \leq s$ , set  $h_j = p^{k-j}f - p^{s-j}ue$ . In  $A$  one obtains

$$R(p^j e) = rh_{s-j}, \quad R(h_j) = up^{s-j}e. \quad (17)$$

The first equality uses  $n \mid ru - 1$ . For the second, the remaining term is  $(1 - ru)p^{k-j}f$ ; using  $nf = 2p^sue$ , it is zero because  $j \leq s$ . Therefore  $R(L_{p^j}) = L_{p^{s-j}}$ . If  $t = 0$ , instead take  $s = k$  and  $R(e) = f$ ,  $R(f) = e$ ; it has the same action  $j \mapsto k - j$ .

By (11),  $L_{p^j}$  is cyclic precisely when  $\min(j, s - j) = 0$ . Hence  $\langle e \rangle = L_1$  is the only cyclic Lagrangian if  $s = 0$ , and there are exactly two if  $s > 0$ . An isometry must send  $\langle e \rangle$  to a cyclic Lagrangian. If  $s = 0$  all isometries are  $S_a$ . If  $s > 0$ ,  $R$  interchanges the two possibilities, so every isometry is  $S_a$  or  $S_a R$ . This proves that the construction exhausts the orthogonal group.

For  $0 < s < k$ , the  $f$  coefficient of  $RS_a R(f)$  modulo  $\langle e \rangle$  is

$$a + rp^{k-s}b_a \equiv a^{-1} \pmod{n},$$

by (14) and  $ru \equiv 1 \pmod{n}$ . Since this element stabilizes  $\langle e \rangle$ , uniqueness in (15) gives  $RS_a R = S_{a^{-1}}$ . For  $t = 0$  the same identity follows by interchanging  $e, f$ . This proves the semidirect product (3) in every case.

## 5 Orbits, orders, and examples

The local action is generated by  $j \mapsto s - j$ . Its invariant is  $\min(j, s - j)$ , with values  $0, \dots, \lfloor s/2 \rfloor$ . Taking products proves (4)–(6) and the transitivity assertion of Theorem 1.1. Formula (11) also shows that the abstract group type of a Lagrangian is a complete orbit invariant in this family. This last statement is not asserted for arbitrary metric groups.

**Corollary 5.1.** *If  $s = k$  then  $|U_{p,k,s}| = \varphi(p^k)$ . If  $s < k$  and  $m = k - s$ , then*

$$|U_{p,k,s}| = p^s \begin{cases} 2, & p \text{ odd}, \\ 1, & p = 2, m = 1, \\ 2, & p = 2, m = 2, \\ 4, & p = 2, m \geq 3. \end{cases}$$

*Proof.* Each unit root of  $a^2 = 1$  modulo  $p^m$  has  $p^s$  unit lifts modulo  $p^k$ . For odd  $p$ , the two roots follow by factoring  $(a - 1)(a + 1)$ , whose factors are coprime at  $p$ . For  $p = 2$ , the two even factors have gcd 2, giving respectively one, two, or four roots for  $m = 1$ ,  $m = 2$ , or  $m \geq 3$ .  $\square$

For example, let  $n = 8$  and  $t = 4$ . Then  $s = 2$  and  $U = (\mathbb{Z}/8\mathbb{Z})^\times \cong C_2^2$ , so  $\text{BrPic} \cong C_2^3$ . The three modules have labels 1, 2, 4 and ranks 8, 4, 2. The two orbits are  $\{M_1, M_4\}$  and  $\{M_2\}$ , with stabilizer orders 4 and 8. The corresponding Lagrangians have types  $C_8, C_2 \times C_4, C_8$ ; the noncyclic one is fixed.

For  $n = 9, t = 3$ , the unit subgroup has order 6 and is cyclic. The Brauer–Picard group is its inversion semidirect product with  $C_2$ , of order 12. Here  $g = 3$  is squarefree, so the two module categories form a single orbit. In contrast, for  $n = 9, t = 0$  the labels 1, 3, 9 form two orbits. Thus the associator can change the action even when the group order is unchanged.

## Exact finite checks

The unrestricted proof above is accompanied by two standard-library Python programs. The first enumerates all possible isometries from the images of  $e, f$ , independently of the displayed generators, and checks the predicted groups and orbits in 383 pairs  $(n, t)$ : all twists for  $2 \leq n \leq 24$  and  $n = 25, 27, 32$ . The second performs 1,556,863 exact checks in each of normal and optimized

Python. It checks the cocycle pentagon, projective and tensor half-braidings, restrictions to subgroups, all small Lagrangian subgroups, and the primary decomposition. Its domains are  $2 \leq n \leq 12$  for cocycles and Lagrangians and  $2 \leq n \leq 24$  for the primary decomposition, with every twist. The tensor-character checks use charges 0, 1, with all fluxes; they are not advertised as an exhaustive test of every charge. The archive records the precise domains and check counts.

The first program uses assertions and is run without optimization; the second uses explicit exceptions and is also tested with `python -O`. Finite checks do not replace the proof or constitute proof-assistant verification. AI-assisted tools were used for derivation, source comparison, computation, and manuscript preparation. This preprint has not undergone independent peer review.

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