

Corner Itineraries and Resonance of ASM Superpromotion

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Abstract

We construct an explicit binary-word factor of Striker–Williams superpromotion on alternating sign matrices of order n . The membership indicator of a specified corner of the ASM poset returns after $3n - 2$ steps. Its itinerary of that length therefore intertwines superpromotion with cyclic rotation. For $n \geq 3$, rotation on the image has effective order exactly $3n - 2$. The proof combines a face-stratum bijection with a time-shift recombination on the tetrahedral poset; it does not enumerate full orbits. The factor can be evaluated with $O(n^4)$ elementary toggle tests. This is a dynamical realization of the proposed resonance frequency, not a static noncrossing model or a characterization of all image words. No full-action periodicity or absolute-priority claim is made.

1 The factor and its scope

Striker and Williams introduced superpromotion on alternating sign matrices (ASMs) through a product of toggles on a tetrahedral poset [1, Section 8]. Their empty-ideal orbit and small orbit census distinguish the frequency $3n - 2$, although the action itself need not have that order. The AIM workshop question [3, Problem 2.7] and the later formulation of Dilks, Pechenik and Striker [2, Problem 5.6] ask for a natural combinatorial factor explaining this frequency. The present paper gives a short corner itinerary and proves its periodicity uniformly in n .

We use the order-ideal model of ASMs, with all conventions specified below. For $n \geq 2$, set $r = n - 2$ and let

$$P_r = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 : a + b + c + d = r\}.$$

Its lower covers are obtained by moving one unit from either b or d to either a or c . Write $\mathcal{J}(P_r)$ for its order ideals and S for superpromotion. For $x \in \mathcal{J}(P_r)$, also write $x_v \in \{0, 1\}$ for the membership indicator of v . In particular, put

$$C(x) = x_{(0,0,r,0)}, \quad B(x) = x_{(0,r,0,0)}, \quad A(x) = x_{(r,0,0,0)}, \quad D(x) = x_{(0,0,0,r)}.$$

Theorem 1.1. *For every $n \geq 2$ and every $x \in \mathcal{J}(P_{n-2})$,*

$$C(S^{3n-2}x) = C(x). \tag{1}$$

Let $N = 3n - 2$ and define

$$f_n(x) = (C(x), C(Sx), \dots, C(S^{N-1}x)), \quad Y_n = f_n(\mathcal{J}(P_{n-2})) \subseteq \{0, 1\}^N. \tag{2}$$

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Then Y_n is invariant under left cyclic rotation and $f_n(Sx) = \text{rot}(f_n(x))$. If $n \geq 3$, the action of rotation on Y_n has effective order exactly N . If $n = 2$, its effective order is 2. The map f_n is computable using $O(n^4)$ elementary toggle tests.

The construction is not an arbitrary quotient obtained by first finding and labeling complete orbits: it uses one fixed corner and a linear number of iterates. Nevertheless, Y_n is defined as a dynamical image. We do not give a static noncrossing diagram, an independent description of its words, or the order-polytope projection suggested in [3]. These stronger geometric goals should not be confused with the explicit factor theorem. In particular, we do not regard this theorem alone as closing every intended interpretation of the original open-ended problem.

2 Coordinates and the toggle word

The root-layer vertex (k, i, j) , representing $e_i - e_j$ in layer $\Phi^+(A_k)$ of [1, Definition 8.4], is related to our coordinates by

$$(k, i, j) = (n - 1 - d, a + 1, a + b + 2), \quad (a, b, c, d) = (i - 1, j - i - 1, k + 1 - j, n - 1 - k).$$

This identifies the two posets. Explicitly, the lower covers of $v = (a, b, c, d)$ are the applicable members of

$$\begin{aligned} L_{BA} &= (a + 1, b - 1, c, d), & L_{BC} &= (a, b - 1, c + 1, d), \\ L_{DA} &= (a + 1, b, c, d - 1), & L_{DC} &= (a, b, c + 1, d - 1). \end{aligned} \tag{3}$$

A member with a negative coordinate is omitted. The rank is $b + d$; lower indicators are at least upper indicators.

The toggle at v adds or removes v when the result is still an ideal, and otherwise does nothing. In indicator notation it replaces x_v by

$$\min_{w \leq v} x_w + \max_{w \geq v} x_w - x_v, \quad \min \emptyset = 1, \quad \max \emptyset = 0. \tag{4}$$

Neighbor values are taken at the instant of the toggle. Each toggle is an involution on ideals.

We evaluate products rightmost first in the convention of [1, Definition 8.14]: S visits decreasing layer k , and within each layer increasing column $i + j$. Equivalently, it visits increasing d , then increasing $2a + b$. Ties are harmless because the tied toggles commute. Of the lower covers in (3), L_{BA} is visited after v , and the other three before v . Of the upper covers, only the move $A \rightarrow B$, namely $(a - 1, b + 1, c, d)$, is visited before v .

The same operator is obtained by visiting increasing clock value

$$\tau(a, b, c, d) = 2a + b + 3d. \tag{5}$$

Indeed, every cover has the same relative order in both words. The corresponding orientations of the cover graph define a common partial order on the toggles. Two linear extensions differ by swaps of adjacent incomparable vertices, whose toggles commute. This justifies the clock description without any periodicity assumption.

3 A face-stratum bijection

Lemma 3.1. *If $x \in \mathcal{J}(P_r)$ vanishes on the face $a = 0$, then $y = Sx$ is given by*

$$y_{(a,b,c,d)} = \begin{cases} 1, & b = 0, \\ x_{(a+1,b-1,c,d)}, & b > 0. \end{cases} \tag{6}$$

Moreover, S maps this stratum bijectively onto the ideals equal to 1 on the face $b = 0$.

Proof. Induct along the toggle word, using the proposed y values at vertices already visited and old x values elsewhere. Suppose first that $b > 0$ and write $z = (a + 1, b - 1, c, d)$. The lower cover $L_{BA} = z$ has not yet been toggled, so its value is x_z . Every other lower-cover value is at least x_z . For L_{BC} , the new value is 1 if $b = 1$; otherwise it is x at a lower cover of z . The new values at L_{DA} and L_{DC} , when present, are likewise old values at lower covers of z . Thus the lower minimum in (4) is exactly x_z .

If $a > 0$, the earlier upper cover $(a - 1, b + 1, c, d)$ has new value x_v by the inductive formula. Every other upper cover has its old value, at most x_v . The upper maximum is therefore x_v . If $a = 0$, the hypothesis gives $x_v = 0$, the specified upper cover is absent, and all remaining old upper values are zero. The same maximum formula holds, including the empty-set default. The toggle gives $x_z + x_v - x_v = x_z$.

If $b = 0$, all lower covers are of type DA or DC , with proposed new value 1. The default lower minimum is also 1. The preceding upper-maximum argument is unchanged, so the toggle gives 1. This proves (6).

For surjectivity, let y be any ideal equal to 1 on $b = 0$ and define

$$x_{(a,b,c,d)} = \begin{cases} 0, & a = 0, \\ y_{(a-1,b+1,c,d)}, & a > 0. \end{cases} \quad (7)$$

If $x_v = 1$, then $a > 0$. Moving to a lower cover never decreases a , and translating that cover by $(-1, 1, 0, 0)$ gives a valid lower cover of the corresponding y vertex. Hence every lower cover also has x value 1, proving that x is an ideal. Formula (6) recovers y , so the stratum map is onto. Injectivity follows from the bijectivity of S . The case $r = 0$ is the single toggle $0 \leftrightarrow 1$. $\square$

Corollary 3.2. *For every ideal x , one has $D(Sx) = 1 - C(x)$.*

Proof. The principal filter of $(0, 0, r, 0)$ is the face $a = 0$, so $C(x) = 0$ means precisely that this face is empty. The principal ideal of $(0, 0, 0, r)$ is the face $b = 0$, so $D(y) = 1$ means precisely that this face is full. Lemma 3.1 identifies these strata exactly. Because S is bijective, it also identifies their complements. Since the indicators are binary, the stated identity follows. $\square$

4 Time-shift recombination

Define the order-reversing automorphism

$$\rho(a, b, c, d) = (b, c, d, a).$$

For any bi-infinite orbit $x(t) = S^t x(0)$, set

$$y_v(t) = 1 - x_{\rho(v)}(t + d(v)). \quad (8)$$

Here $d(v)$ denotes the fourth coordinate of v .

Lemma 4.1. *The configurations $y(t)$ in (8) form a superpromotion orbit. The map $R : x(0) \mapsto y(0)$ is a bijection of $\mathcal{J}(P_r)$ satisfying*

$$RS = SR, \quad R^4 = S^r. \quad (9)$$

Proof. For adjacent vertices define $\epsilon(v, w) = 1$ if w is toggled before v in one S step, and 0 otherwise. The processing order gives

$$d(w) - d(v) = \epsilon(\rho(v), \rho(w)) - \epsilon(v, w). \quad (10)$$

For completeness, when v is lower and w upper, all cases are:

Move $v \rightarrow w$	$d(w) - d(v)$	$\epsilon(v, w)$	$\epsilon(\rho(v), \rho(w))$
$A \rightarrow B$	0	1	1
$C \rightarrow B$	0	0	0
$A \rightarrow D$	1	0	1
$C \rightarrow D$	1	0	1

Reversing an edge reverses the identity as well.

First verify that $y(t)$ is an ideal. For a lower-to-upper edge with equal d coordinates, this follows from order reversal by ρ and complementation. In the other case, the upper vertex w has $d(w) = d(v) + 1$. The vertex $\rho(w)$ is lower than $\rho(v)$ and is toggled earlier. Immediately after that lower toggle and before the upper toggle, the configuration is still an ideal. Therefore, with $u = t + d(v)$,

$$x_{\rho(w)}(u + 1) \geq x_{\rho(v)}(u).$$

Taking complements gives $y_v(t) \geq y_w(t)$, as required.

The sequential toggle recurrence can be written

$$x_v(t + 1) + x_v(t) = \min_{w < v} x_w(t + \epsilon(v, w)) + \max_{w > v} x_w(t + \epsilon(v, w)). \quad (11)$$

Substitute (8). Complementation exchanges lower minima and upper maxima; ρ exchanges lower and upper covers. The defaults 1 and 0 are exchanged too. Identity (10) aligns all the time arguments, so (11) for y is exactly the recurrence for x at $\rho(v)$, with both sides complemented about 2. The recurrence uniquely implements the fixed sequential toggle word. Thus $y(t + 1) = Sy(t)$, and $RS = SR$.

Now apply (8) four times. There are four complements, $\rho^4 v = v$, and the accumulated time shift is

$$d(v) + d(\rho(v)) + d(\rho^2(v)) + d(\rho^3(v)) = r.$$

Hence $R^4 = S^r$. Since S^r is a bijection, so is R . No proposed frequency or full-action periodicity was assumed in this argument. $\square$

5 Corner identities and faithful rotation

At the four corners, the definition of R reads

$$\begin{aligned} C(Rx) &= 1 - B(x), & B(Rx) &= 1 - A(x), \\ A(Rx) &= 1 - D(x), & D(Rx) &= 1 - C(S^r x). \end{aligned} \quad (12)$$

Lemma 5.1. *Writing $q = r + 1 = n - 1$, for every ideal x one has*

$$C(S^q x) = 1 - B(x), \quad B(S^q x) = 1 - A(x), \quad A(S^q x) = 1 - D(x). \quad (13)$$

Consequently $C(S^{3r+4} x) = C(x)$.

Proof. Apply Corollary 3.2 to Rx and use $RS = SR$. By (12), the left side is $1 - C(S^{r+1}x)$ and the right side is $B(x)$. This proves the first identity in (13). Apply it to Rx and again use (12); the result is the second identity. Repeating this argument gives the third. Thus

$$C(S^{2q}x) = A(x), \quad C(S^{3q}x) = 1 - D(x).$$

Finally apply the latter identity to Sx and use $D(Sx) = 1 - C(x)$. Since $3q + 1 = 3r + 4$, this proves the return. $\square$

Lemma 5.2. *For $0 \leq h \leq r$, set $J_h = \{v : b(v) + d(v) \leq h\}$ and let $J_{-1} = \emptyset$. Then $S(J_h) = J_{h-1}$.*

Proof. Induct along the word and write $p = b + d$ for the rank of the vertex being toggled. If $p < h$, then its lower values are all 1. Moreover $a + c = r - p > 0$, so an upper cover obtained by moving A or C to D exists. It is visited later and still has old value 1, because its rank is $p + 1 \leq h$. Thus the upper maximum is 1, and the vertex stays 1.

If $p = h$, lower values are 1 and upper values 0, with the same empty-set defaults, so the vertex changes from 1 to 0. If $p > h$, upper values are 0. There is an earlier lower cover of new value 0: take L_{BC} if $b > 0$, and L_{DA} otherwise, when $d > 0$. Its rank is $p - 1 \geq h$. Hence the vertex stays 0. These are exactly the indicators of J_{h-1} . $\square$

Proof of Theorem 1.1. Lemma 5.1 proves (1). It follows directly that the itinerary (2) rotates one place under S , so Y_n is rotation-invariant and f_n is an equivariant surjection. Rotation has order dividing N on this image.

To prove its effective order, let $F = P_r = J_r$ be the full ideal. Lemma 5.2 gives $S^t F = J_{r-t}$ for $0 \leq t \leq r + 1$. Therefore $C(S^t F)$ and $A(S^t F)$ are 1 for $0 \leq t \leq r$ and 0 at $t = r + 1$, while $B(F) = 1$ and $B(S^t F) = 0$ for $1 \leq t \leq r + 1$. Apply (13) and $C(S^{2r+2}x) = A(x)$. Together these cover the whole length- N word and show that $f_n(F)$ has exactly two zeros, at indices

$$r + 1 \quad \text{and} \quad 3r + 3,$$

where indices start at zero. Their circular gaps are $2r + 2$ and $r + 2$. When $r > 0$ these are unequal. A nonidentity rotation stabilizing a two-element set must interchange its elements, which would force equal gaps. Thus $f_n(F)$ has period exactly N , proving faithfulness. At $r = 0$ there is one poset vertex, S is a two-cycle, and the words are 1010 and 0101, of period 2.

Finally, P_r has $\binom{n+1}{3}$ vertices and each has at most four lower and four upper covers. With adjacency lists, a toggle uses a bounded number of membership tests. Computing N steps and recording one bit after each therefore takes $O(n^4)$ elementary tests. $\square$

6 Limitations and reproducibility

The four corner observables all return after $3n - 2$ steps, by the same identities. This statement is weaker than periodicity of the whole configuration: the published census at $n = 7$ includes length-57 orbits, although $3n - 2 = 19$ [1, Figure 22]. Return of every boundary-face coordinate is a separate experimental observation and is not proved here.

The Boolean hypothesis also matters. Extend (4) to integer labels in $[0, K]$, using lower default K , preserving the same order inequalities. At $n = 3$, $K = 2$, order the labels as (C, A, D, B) . Starting at $(2, 2, 1, 1)$, seven steps give $(1, 2, 1, 1)$, so even the corner does not return. This rules out that naive unrestricted piecewise-linear extension, not the theorem on order ideals. Similarly, recording the raw boundary of an ideal need not determine the boundary after a step; a finite

$n = 6$ witness is retained in the supplementary checks. The itinerary is not being replaced by that projection.

The accompanying standard-library Python package reproduces the ASM orbit census through $n = 7$, checks all four corner identities on all ideals through that size, and tests the face-transfer formula on every face-stratum ideal through $n = 8$. The corner tests comprise 905,056 exact assertions. Recombination is checked exhaustively through $n = 5$ and on deterministic, nonuniform samples through $n = 10$. Rank descent and the two-zero full-ideal word are checked separately through $n = 30$. These finite calculations are regression evidence; the arbitrary- n result rests on the preceding proofs. No proof assistant was used.

The ASM poset, superpromotion, and the empty/full-ideal orbit calculation are credited to [1]; resonance and binary-content rotation in related systems are treated in [2]. Time-shift recombination and itinerary coding are used as methods, not claimed as newly invented general principles. A bounded primary-source literature search found no matching corner-factor proof but does not certify absolute priority.

This manuscript was developed with AI assistance for mathematical exploration, computation, drafting and self-audit. It is an unrefereed preprint; the audit is not independent peer review. The precise claimed result is Theorem 1.1, with the source-scope limitations stated in Section 1.

References

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