Full Linear Homomesy Spaces for Rowmotion on Rooted Forests

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Abstract

For every finite rooted forest with minimal roots, we compute the full spaces of linear order-ideal and antichain statistics that have zero average on every rowmotion orbit. The algorithm retains the orbit through the empty ideal and two affine hulls, each represented by at most $n+1$ vectors, instead of enumerating the orbits. A deletion lemma for Cartesian products yields scalar recursions for both zero-mesy dimensions and an exact rational basis algorithm using $O(n^4)$ field operations, apart from integer gcd/lcm operations. The two full homomesy dimensions agree; the zero-mesy dimensions can differ by one. A known acyclic-toggle theorem transfers the ideal-indicator result to Coxeter toggle actions. The rooted-tree orbit construction is established prior work. Our contribution is its affine compression and the resulting complete linear-space calculation, not a solution for arbitrary posets.

1 The spaces and the main formula

Let P be a finite poset and $\mathcal{J}(P)$ its lower order ideals. Rowmotion is the permutation

$$\rho(I) = \downarrow\min(P \setminus I).$$

For $x \in P$, consider the ideal and antichain indicators $v_I(I)_x = \mathbf{1}_{x\in I}$ and $v_A(I)_x = \mathbf{1}_{x\in \max I}$. The latter uses the usual bijection between ideals and antichains. For a rowmotion orbit O, define its two normalized average vectors

$$a_s(O) = \frac{1}{|O|}\sum_{I\in O} v_s(I), \qquad s \in \{I, A\}.$$

Write $\mathcal{Z}_s(P)$ for coefficient vectors $c \in \mathbb{R}^P$ satisfying $c \cdot a_s(O) = 0$ for every orbit. Write $\mathcal{H}_s(P)$ for the vectors whose value $c \cdot a_s(O)$ is independent of O, with no prescribed mean. These coefficient descriptions faithfully describe the corresponding statistic spaces: ideal indicators are independent by evaluation on principal ideals in a linear extension, and antichain indicators by evaluation on singleton antichains.

Problem 1.1 in the 2015 AIM workshop list [1] asks for dimensions and natural bases of such zero-mesy spaces, for general posets and actions, emphasizing rowmotion and promotion. We address only rooted forests: each connected Hasse component is a tree with a unique minimal element. A fixed vertex order will give a reproducible elimination basis; we do not claim an automorphism-invariant notion of a natural basis.

Dangwal et al. [2] develop rowmotion on rooted trees through tilings, component-cycle products and root-branch insertion, and obtain coordinate orbit counts and results for several families. We use this

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established orbit mechanism, but compress the averages rather than list all cycles. The distinction from toggleability spaces is important: those require a statistic to be a constant plus a linear combination of toggleability statistics; recent dimension results in [3] concern that more restrictive certificate space on other poset families. Here we compute the entire linear homomesy spaces for rooted forests.

For each rooted tree T, define integers $L(T), d(T)$ and Boolean values $u(T), e(T)$ recursively. If $T_1, \dots, T_k$ are the components above its root, put

$$\ell = \operatorname{lcm}_i L(T_i), \qquad g = \frac{\prod_i L(T_i)}{\ell}, \qquad q = \#\{i : u(T_i) = 0\}, \tag{1}$$

$$L(T) = \ell + 1, \qquad u(T) = \mathbf{1}_{g=1,\, q=0}, \qquad e(T) = \mathbf{1}_{g=1,\, q\le 1}, \tag{2}$$

$$d(T) = \sum_i d(T_i) + 1 - e(T). \tag{3}$$

Empty products and lcm are one, and empty sums are zero. Thus a leaf has $(L, u, e, d) = (2, 1, 1, 0)$.

Theorem 1. *Let $F = T_1 \sqcup \cdots \sqcup T_k$ be a nonempty rooted forest with n vertices, and set $D = \sum_i d(T_i)$ and $E = \max_i e(T_i)$. Then*

$$\dim \mathcal{H}_I(F) = \dim \mathcal{H}_A(F) = n - D, \tag{4}$$

$$\dim \mathcal{Z}_I(F) = n - D - E, \qquad \dim \mathcal{Z}_A(F) = n - D - 1. \tag{5}$$

The algorithm of Section 3 constructs rational bases of all four spaces without enumerating ideals or orbits. It uses $O(n^4)$ rational field operations, apart from integer gcd/lcm operations. All four spaces are zero-dimensional for the empty forest.

This is a scoped answer for ordinary rowmotion. It does not classify arbitrary posets, arbitrary invertible actions, or piecewise-linear and birational rowmotion. The frozen research aid for AIM-OTHER-0001 in [5] already contains the orbit-matrix formulation and bases for unions of rowmotion-transitive components; we credit that starting point. The extension here is to all rooted forests. Bounded literature searches do not exclude folklore or differently indexed prior results, and no absolute-priority claim is made.

2 The marked orbit and deletion lemma

Distinguish the *marked orbit* $O^0(P)$ containing the empty ideal. Let $L(P) = |O^0(P)|$, and let $u(P) = 1$ precisely when it is the only orbit. For each feature s, retain its marked average $a_s^0(P)$ and the affine hulls $H_s(P)$ of all averages and $H_s^-(P)$ after removing just that one orbit. The deletion is of an orbit, not of a distinct vector value. Use aff $\varnothing = \varnothing$.

Lemma 2 (Root insertion)**.** *Let $T = \{r\} \oplus F$ be obtained by putting a new minimal root below a rooted forest F. Its marked length is $L(F) + 1$ and $u(T) = u(F)$. With $\sigma = L(F)/(L(F) + 1)$ its marked averages are*

$$a_I^0(T) = \sigma(1, a_I^0(F)), \qquad a_A^0(T) = (1/(L(F) + 1), \sigma a_A^0(F)). \tag{6}$$

For every other orbit the average transforms by

$$a_I \longmapsto (1, a_I), \qquad a_A \longmapsto (0, a_A). \tag{7}$$

For both features the marked tree average is outside the affine hull of the others. In addition, the ideal-average row rank of T is $1 + \dim H_I(F)$.

Proof. Every nonempty ideal of T is $\{r\} \cup J$ for $J \in \mathcal{J}(F)$. If $J \neq F$, its image is $\{r\} \cup \rho_F(J)$. The full ideal of T maps to empty, and empty maps to $\{r\}$. Thus precisely the transition $F \to \varnothing$ is subdivided by one extra state; other cycles lift unchanged. This also holds for empty F.

The root occurs in every lifted nonempty ideal and in no empty ideal, giving the ideal formulas. For maximal antichains, child-coordinate totals are unchanged; the lift of the empty F-ideal contributes $\{r\}$, and the extra empty T-ideal contributes zero. This proves the antichain formulas. The root coordinate of the marked average is different from the common root coordinate of all others, proving affine essentiality. Finally the span of all ideal averages on T equals the span of $(1, a_I(O))$ over all F-orbits: the marked row has merely been rescaled by nonzero σ. The augmented-row rank is $1 + \dim H_I(F)$. □

Lemma 3 (Deleting a marked product orbit)**.** *Let $F = T_1 \sqcup \cdots \sqcup T_k$, with marked component lengths L_i, and define ℓ, g, q by (1). Then $L(F) = \ell$, $a_s^0(F) = (a_s^0(T_1), \ldots, a_s^0(T_k))$, and*

$$H_s(F) = \prod_i H_s(T_i), \qquad u(F) = \mathbf{1}_{g=1,\, q=0}.$$

Its deleted affine hull is determined by these four cases:

1. *If $g > 1$, then $H_s^-(F) = H_s(F)$.*
2. *If $q \geq 2$, the same equality holds, including when $g = 1$.*
3. *If $g = 1, q = 0$, then $H_s^-(F) = \varnothing$.*
4. *If $g = 1, q = 1$, let j be the unique nontransitive component. Then $H_s^-(F)$ has block $H_s^-(T_j)$ in position j, and each other block fixed at $a_s^0(T_i)$.*

Proof. Rowmotion acts componentwise. Component cycles of lengths ℓ_i produce $\prod_i \ell_i / \operatorname{lcm}_i \ell_i$ cycles. Every component state occurs equally often in each such cycle, so its normalized average is the concatenation of the component averages, independent of phase. This proves the full-hull, marked-length and transitivity assertions.

If $g > 1$, a different phase cycle retains the marked average after deletion. If $q \geq 2$, choose an alternative orbit in each of two components, keeping the other components marked. The resulting three nonmarked tuples have averages satisfying the rectangle identity

$$a_{00} = a_{10} + a_{01} - a_{11},$$

where a_{00} is the all-marked average. Each tuple gives at least one cycle, proving case 2 even if some vector values coincide. Case 3 is transitivity. In case 4 the all-marked tuple has just one cycle; every remaining tuple varies precisely the nontransitive component. Taking its affine hull gives the stated product with fixed blocks. These arguments do not assume distinct orbits have distinct averages. □

The cycle mechanism behind Lemma 2 is classical in the rooted-tree setting; see the outer-product discussion and Theorem 4.1 of [2]. The useful compression step is retaining the deleted affine hull, for which Lemma 3 needs only the marked lengths and the transitivity flags.

3 The compression algorithm

Process the rooted forest from leaves upward. Store

$$L(P), \quad u(P), \quad a_s^0(P), \quad B_s(P), \quad B_s^-(P),$$

where the last two lists are affine-independent generators of $H_s(P)$ and $H_s^-(P)$, respectively. Store these separately for $s = I, A$. The empty forest has marked vector $()$, length one, $u = 1$, full list $[()]$, and empty deleted list.

For a union, concatenate the marked vectors to obtain b. Generate its full affine hull by b and the vectors obtained by replacing a single component block of b by a row of that component's full list. Indeed, their differences span the direct sum of the component direction spaces. For the deleted hull, apply the four cases of Lemma 3: reuse the full list, use an empty list, or substitute the single deleted component list into the fixed marked blocks. At a root apply (7) to the deleted list and adjoin (6) for the full list. Select an affine basis by Gaussian elimination on the augmented rows $(1, a)$, retaining independent original rows.

Affine maps commute with taking affine hulls. Induction using the two lemmas therefore proves that every retained hull is exact. A retained generator need not be the average of an actual surviving orbit; the algorithm uses its affine-span property, not such an interpretation. In m coordinates each list contains at most $m+1$ rows. At a union, the number of candidate rows is at most $1+\sum_i(n_i+1) = O(m)$; at a root it is again $O(m)$. Incremental elimination takes $O(m^3)$ field operations. Summing this bound over $O(n)$ tree and union nodes gives $O(n^4)$. This is a coarse arithmetic-operation bound, not a sharp bit-complexity estimate.

Let M_s have the final full-list rows. Equal affine hulls have equal linear spans, hence

$$\mathcal{Z}_s(F) = \ker M_s.$$

For an explicit rational basis, reduce M_s to row-echelon form. For each free column, set its coefficient to one, the other free coefficients to zero, and solve the pivot equations. This produces independent vectors spanning the kernel; denominators may be cleared. The same construction on rows $a - a_s^0(F)$ gives $\mathcal{H}_s(F)$ and its common mean is $c \cdot a_s^0(F)$. Fixing vertex order and pivot choices makes this procedure deterministic. No ideal or orbit enumeration occurs.

4 Proof of the dimension formulas

We first prove by induction that, for either feature,

$$\dim H_s(T) = d(T).$$

Suppose the child dimensions are known, and put $d_F = \sum_i d(T_i)$. If $g = 1, q = 0$, all components and the new tree have one orbit, so $d_F = d(T) = 0$. If $g = 1, q = 1$, deleting the marked average of the single nontransitive child reduces its affine dimension by exactly one, because Lemma 2 makes it affinely essential. The deleted forest hull then has dimension $d_F - 1$. In the remaining cases its dimension is d_F, by Lemma 3. Root insertion adjoins one affinely essential marked point to these nonempty deleted hulls. In every case the resulting dimension is $d_F + 1 - e(T)$, as claimed.

For ideal membership, Lemma 2 gives row rank

$$r_I(T) = 1 + d_F = d(T) + e(T).$$

For a nonempty affine hull its linear rank is its affine dimension plus one precisely when the origin is outside it. Thus $e(T)$ records whether $0 \notin H_I(T)$. On a forest the affine dimensions add, and the Cartesian product contains zero if and only if every factor does. Consequently its row rank is $D + E$, proving the ideal formula.

For antichains there is always a strictly positive coefficient vector with a nonzero common orbit average. Normalize the vector on a tree to have average one as follows. A leaf receives coefficient 2. At a root with $k > 0$ children, assign the root coefficient 1 and divide the normalized coefficients of each child by k. By induction the child-forest statistic has average one on every orbit. Its average on an unmarked lifted orbit stays one, while the marked average becomes

$$\frac{1}{L(F)+1}+\frac{L(F)}{L(F)+1}=1$$

by (6). This proves the construction. Concatenate these vectors over the k components of a nonempty forest; its mean is $k>0$, so $0\notin H_A(F)$. The antichain row rank is therefore $D+1$. Finally, in both features the direction space of the average hull has dimension D, and its annihilator is the full homomesy space. This proves all of Theorem 1.

5 Examples, toggle actions and verification

For a disjoint union of chains every component is transitive with $d=0, e=1$. Both zero-mesy spaces have codimension one, recovering the elementary union-of-chains case. For the three-element tree with minimal root and two leaves, the ideal average rows are

$$(2/3,1/3,1/3),\quad (1,1/2,1/2),$$

and the antichain rows are

$$(1/3,1/3,1/3),\quad (0,1/2,1/2).$$

The ideal zero-mesy space has basis $(-1,2,0),(-1,0,2)$; the antichain one has basis $(0,1,-1)$. Thus the equality of full homomesy dimensions does not identify the two zero-mesy spaces.

A star with 100 leaves has 2^{99} rowmotion cycles: deleting the root gives 100 two-cycles acting in parallel, and root insertion does not change their number. Nevertheless two affine rows suffice for either feature. The algorithm handles this example without enumerating its ideals or cycles.

Corollary 4 (Attributed Coxeter transfer)**.** *The ideal-indicator spaces and means computed above are unchanged if rowmotion on a rooted forest is replaced by any product in which each ideal toggle occurs exactly once.*

Proof. The noncommutation graph of ideal toggles is a subgraph of the Hasse graph and is therefore acyclic. Apply the indicator-preserving Coxeter-toggle theorem of Elizalde–Plante–Roby–Sagan [4, Theorems 5.1–5.3]. Rowmotion itself is such a product. ☐

We pin the preprint version because the published paper has different Section 5 numbering. This corollary is an application of a known theorem, not a new transfer principle. It covers any promotion convention of the specified ideal-toggle form; no assertion is made about an unspecified antichain promotion.

The accompanying standard-library checker compares the compressed full and deleted affine hulls, marked lengths and rows, transitivity flags, dimensions and nullspace bases to independent bit-mask orbit enumeration. It exhausts all 3,047 unlabelled rooted forests with at most ten vertices, including the empty forest, covering 275,986 ideals and 47,043 cycles. It also checks all toggle orders for forests with at most five vertices, totaling 2,646 orders. An additional checker tests the positive antichain statistic and its rank formula on the same forests, as well as nonenumerative large-star and chain examples. There are 731,730 and 53,141 exact assertions, respectively, reproduced in isolated Python 3.9 and 3.13 runs. These are diagnostic tests, not a formal proof or independent peer review; the general argument is the induction given above.

This unrefereed manuscript was developed with AI assistance for derivation, literature search, code and exposition. Its proof audit is an originating-researcher self-audit. Known orbit mechanisms and the Coxeter transfer are credited explicitly; the novelty of the affine compression is not independently certified.

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