Sharp Computability Bounds for Nonforking Sections

Alper Ferudun*

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Abstract

We study computability of sections of the restriction map $S_1(N) \to S_1(M)$ for countable stable models $M \preccurlyeq N$. Types are represented by characteristic functions on formulas with named parameters, and the two elementary diagrams are separately decidable. Classical stable definability gives a continuous section computable from the input type together with the fixed oracle $0'$. This bound is sharp, already for the decidable, ω-stable, ω-categorical theory of infinitely many infinite equivalence classes. For a fixed decidable-range elementary inclusion in this theory, an oracle X computes a section exactly when it computes the set of classes of N that meet M. Explicit inclusions realize every computably enumerable degree as this least auxiliary degree. Thus a computable section need not exist, although every individual type in the example is computable. In contrast, a decidable full diagram of the predicate expansion (N, M) gives a computable section for every stable theory. The result addresses a Type-2 formulation of an AIM question whose printed statement leaves the effective presentation unspecified.

1 The question and the effective conventions

Question 6 in Section 1.3 of the 2013 AIM workshop problem list on computable stability theory asks whether restriction of types between models of a stable theory has a computable section [1]. The discussion, attributed there to Miller, recalls the continuous nonforking section. The printed question does not specify how types or elementary pairs are presented. We state these choices explicitly: with separate decidable diagrams there is a negative example, whereas decidability of the full predicate-pair diagram gives a positive answer. No assertion about every possible representation is intended.

Let L be a computable countable first-order language. A decidable elementary diagram of a countable structure decides all first-order formulas with named parameters from that structure. Given models $M \preccurlyeq N$ and a computable elementary embedding, we identify M with its image when writing formulas. The two diagrams are assumed decidable separately; this assumption does not supply the elementary diagram of (N, P) with $P(N) = M$.

Fix effective enumerations of the one-free-variable formulas over each structure. A type $p \in S_1(M)$ is named by its characteristic function on this enumeration. A section of restriction is a map

$$s : S_1(M) \longrightarrow S_1(N), \qquad s(p){\restriction}M = p.$$

It is *X-computable* if one oracle functional, given the name of any p and the fixed auxiliary oracle X, computes the characteristic function of $s(p)$. Totality is required only on names of actual complete types. The notion is uniform in the input type, not a separate algorithm selected nonuniformly

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`.

for each type. The structures and their presentations are fixed when discussing the least auxiliary Turing degree of a section.

The main statements are the general $0'$ upper bound in Theorem 2.2, the fixed-pair degree criterion in Theorem 3.3, and its realization by every c.e. degree in Theorem 4.1. Stable definability of types is classical; we use the formulation in Ben Yaacov [2, Theorem 3 and Corollary 4]. Related representation-sensitive selection questions are studied by Sepúlveda-Jiménez [3, Section 7.2]. That preprint considers choosing a finite definition from a represented structure and a local type. Our fixed-pair problem asks instead for a single section on all complete types and its least auxiliary degree. We use no result of that preprint as a proof dependency. This note is unrefereed and makes no absolute priority claim for the elementary reductions below.

2 A fixed jump suffices in every stable theory

Lemma 2.1. *For any elementary inclusion $M \preccurlyeq N$, a continuous section of $S_1(N) \to S_1(M)$, if it exists, is unique.*

Proof. The principal types $p_a = \mathrm{tp}(a/M)$, $a \in M$, are dense in $S_1(M)$. Indeed, if a formula $\theta(x, \bar{a})$ belongs to a complete type over M, its existential closure is true in an elementary extension of M, hence in M. Every section must send p_a to $\mathrm{tp}(a/N)$ because $x = a$ belongs to p_a. Two continuous sections therefore agree on a dense subset and, since $S_1(N)$ is Hausdorff, agree everywhere. □

Theorem 2.2. *Let T be a complete stable theory in a computable countable language. Let $M \preccurlyeq N$ be countable models with separately decidable elementary diagrams and a computable elementary embedding. The unique continuous section of restriction is computable from $p \oplus 0'$ on input p. If the full elementary diagram of (N, P), where P names the embedded copy of M, is decidable, the section is computable without an auxiliary oracle. No decidability assumption on the range of the embedding is needed for the first assertion.*

Proof. Fix an output formula $\varphi(x, \bar{b})$ over N. By stable definability of types, applied with object and parameter variables interchanged to $\mathrm{tp}(\bar{b}/M)$, there is a formula $\psi(x, \bar{a})$ over M such that

$$\text{for every } m \in M, \qquad M \models \psi(m, \bar{a}) \quad \Longleftrightarrow \quad N \models \varphi(m, \bar{b}). \tag{1}$$

This is the classical external-trace consequence of stability [2]; it applies also when $\bar{b}$ is a tuple.

Enumerate all candidate formulas $\psi(x, \bar{a})$. Since the diagrams and embedding are computable, condition (1) is a Π^0_1 condition on a candidate, uniformly in φ and its parameters. The fixed oracle $0'$ decides it. Find the first correct candidate and declare

$$\varphi(x, \bar{b}) \in s(p) \quad \Longleftrightarrow \quad \psi(x, \bar{a}) \in p. \tag{2}$$

The search for ψ is independent of p. In particular, the oracle used is $p \oplus 0'$, not the potentially stronger jump p'.

We verify that these coordinate decisions form a type. Two definitions of the same trace agree throughout M, so they are equivalent in its elementary diagram and have the same value in every type over M. Traces preserve complements and finite intersections. If finitely many output formulas are declared true, the conjunction of their trace definitions belongs to p. Its existential closure is true in M, so a witness in M satisfies all the output formulas in N. Hence the proposed set is finitely consistent with the elementary diagram of N. The complement rule makes it complete. For a formula over M, the formula itself defines the trace, proving the section property.

Every output coordinate reads just one fixed formula coordinate of p, after a search independent of p. Thus the section is continuous. Lemma 2.1 gives uniqueness, and consequently identifies it with the usual stable nonforking section.

Finally, if the full elementary diagram of (N, P) is decidable, correctness of a candidate is decided directly by the single first-order statement

$$(N, P) \models \forall x (P(x) \longrightarrow (\varphi(x, \bar{b}) \leftrightarrow \psi(x, \bar{a}))).$$

The same search then requires no jump. □

3 The exact degree for an equivalence relation

Let T_E be the theory, in the language $\{E\}$, of an equivalence relation with infinitely many classes, each infinite. We recall its elementary properties to keep the lower bound explicit.

Lemma 3.1. *The theory T_E is complete, decidable, ω-stable and ω-categorical, with effective quantifier elimination. For any $M \models T_E$, the complete one-types over M are precisely:*

1. $p_a = \mathrm{tp}(a/M)$ *for* $a \in M$*;*
2. p_C*, a new element in a class* C *represented in* M*;*
3. p_∞*, an element in a new class disjoint from* M*.*

If M has a decidable elementary diagram, p_∞ is computable, and p_a is computable uniformly from a.

Proof. To eliminate an existential variable from a Boolean combination of equality and E-atoms, first use disjunctive normal form. An equality with another variable permits substitution. Otherwise, positive E-constraints prescribe a class or negative constraints require avoiding finitely many classes. Consistency is determined by the finite equality/E-pattern of the remaining variables. Any consistent prescribed class supplies a fresh element, and there is always a class outside a finite list. Discard inconsistent patterns and take the finite disjunction. This is an effective procedure; negation also eliminates universal quantifiers. It proves completeness and decidability.

Quantifier elimination gives the stated type classification. The same description over a countable parameter set gives countably many one-types, so T_E is ω-stable. Every countable model consists of countably infinitely many countably infinite classes. Matching classes and then their elements gives ω-categoricity. To decide p_∞, eliminate quantifiers, evaluate parameter-only atoms, and set $x = a$ and $E(x, a)$ false for every $a \in M$; reflexive atoms are true. The elementary diagram decides p_a. □

Lemma 3.2. *If $F \subset M$ is finite and the class of $a \in M$ is disjoint from F, then p_a and p_∞ agree on every formula whose parameters lie in F.*

Proof. They have the same equality and E-pattern over F. Apply quantifier elimination from Lemma 3.1. □

For the rest of this section let $M \preccurlyeq N$ be models of T_E with separately decidable elementary diagrams and a computable elementary embedding with decidable range. Put

$$D(M, N) = \{b \in N : (\exists a \in M)\ E(a, b)\}.$$

Theorem 3.3. *For every oracle X, an X-computable section of $S_1(N) \to S_1(M)$ exists if and only if $D(M,N) \leq_T X$.*

Proof. Suppose first that X decides $D = D(M,N)$. For $b \in D$, find $r(b) \in M$ with $E(r(b),b)$ by search. If $b \in M$, range decidability recognizes this and search through the embedding recovers its M-name. For $p \in S_1(M)$ define atomic answers over N by

$$[x = b]_{s(p)} = \begin{cases} [x = b]_p, & b \in M, \\ 0, & b \notin M, \end{cases}$$

$$[E(x,b)]_{s(p)} = \begin{cases} [E(x, r(b))]_p, & b \in D, \\ 0, & b \notin D. \end{cases}$$

Here $[\theta]_p$ denotes the characteristic bit of θ in p. Parameter-only atoms are evaluated in N, and reflexive atoms are true. Quantifier elimination supplies all other answers. The classification in Lemma 3.1 verifies consistency: p_a extends to the same named element, p_C to a new element of the same class, and p_∞ to a new class over N. Each restriction is p. Representatives do not affect the answer, because E-equivalent parameters give equivalent formulas. This is an X-computable section.

Conversely, suppose a functional Φ with auxiliary oracle X computes some section, not assumed in advance to be the nonforking section. For $b \in N$, run Φ on the computable input p_∞ to answer the output question $E(x,b)$. The run halts. Let F be the finite union of all M-parameters occurring in the input-type queries made during this run. The entire transcript, and hence F, can be computed from X and b.

The returned bit is zero. Indeed, if $b \in D$, choose $a \in M$ with $E(a,b)$. Since $\neg E(x,a) \in p_\infty$, an extension cannot contain $E(x,b)$. If $b \notin D$ and the bit were one, choose $a \in M$ from a class disjoint from F. By Lemma 3.2, the run with input p_a follows the identical transcript and returns one. This identity follows inductively even for adaptive queries. But the section property forces the output at p_a to be $\mathrm{tp}(a/N)$, where $E(a,b)$ is false, a contradiction.

We claim that the recorded finite set satisfies

$$b \in D \iff (\exists c \in F)\ E(b,c). \tag{3}$$

The reverse implication is immediate. If $b \in D$ but its class avoids F, choose $a \in M$ in that class. Lemma 3.2 again gives the same transcript on p_a, so the output bit would be zero. The forced output $\mathrm{tp}(a/N)$ instead contains $E(x,b)$, a contradiction. The finite test in (3) decides D from X. □

The lower reduction uses only the finite use of a terminating computation and the values forced at principal types. It invokes no effective modulus of continuity or compactness oracle and no procedure for recognizing valid type names. All names used in the argument are genuine complete types.

4 Realizing every computably enumerable degree

Theorem 4.1. *For every c.e. set A, there are countable models $M \preccurlyeq N$ of T_E with separately decidable elementary diagrams and a computable decidable-range inclusion such that $D(M,N) \equiv_T A$.*

Proof. Fix a computable increasing sequence of finite sets A_s with union A. Take the domain of N to be a computably coded disjoint union

$$\{m(e,s,k) : e,s,k \in \mathbb{N}\} \ \sqcup \ \{d(i,k) : i,k \in \mathbb{N}\} \ \sqcup \ \{b(e,k) : e,k \in \mathbb{N}\},$$

and let M consist of the m- and d-elements. Use disjoint formal families of labels $B_e, C_{e,s}, D_i$, and assign

$$\ell(m(e,s,k)) = \begin{cases} B_e, & e \in A_s, \\ C_{e,s}, & e \notin A_s, \end{cases}$$
$$\ell(d(i,k)) = D_i, \qquad \ell(b(e,k)) = B_e.$$

Define $E(u,v)$ by equality of labels. This relation is decidable because membership in the fixed finite stage A_s is decidable. Importantly, s is part of the permanent name of an element: an earlier $C_{e,s}$-label does not change when e later enters A.

Every represented class in either structure is infinite, by varying k; the D_i-classes guarantee infinitely many classes in both. Thus both structures satisfy T_E. Their atomic diagrams are decidable, so effective quantifier elimination makes their elementary diagrams decidable. The inclusion is computable with decidable range and preserves the atomic relations, hence is elementary by quantifier elimination.

The class of $b(e,0)$ meets M exactly when $e \in A$: in that case some $m(e,s,0)$ has label B_e, and otherwise no m- or d-element does. This proves $A \leq_T D(M,N)$. Conversely, every m- or d-element already lies in M, while $b(e,k) \in D(M,N)$ exactly when $e \in A$. Hence $D(M,N) \leq_T A$. ☐

Corollary 4.2. *There is a decidable-range elementary inclusion of separately decidable models of a decidable, ω-stable, ω-categorical theory for which no computable section exists. Its least auxiliary section degree is $0'$. Consequently the fixed-jump bound of Theorem 2.2 is sharp.*

Proof. Take A to be the halting set and apply Theorems 3.3 and 4.1. ☐

5 Scope, uniformity, and verification

The obstruction is uniformity, not a noncomputable individual output. In any decidable model of T_E, every individual one-type is computable: Lemma 3.1 treats the principal and new-class types, and p_C can be computed after fixing one representative of C. Such a representative exists but is not given uniformly by a type name. In particular, the output of the canonical section at p_∞ is itself computable. What fails for the halting-coded pair is one functional working on *all* input names.

Nor is the predicate expansion of that pair decidable. The formula

$$(\exists x)\big(P(x) \wedge E(x, b(e,0))\big)$$

decides the halting set. Calling the example a "decidable pair" without the separate-diagrams qualification would therefore be misleading. Theorem 2.2 proves the opposite conclusion under the stronger full-pair input convention. Together these statements supply a complete answer under the conventions fixed here, while preserving the ambiguity of the original workshop formulation.

A finite regression checker accompanies the source. It considers all $3^4 = 81$ schedules in which four synthetic indices enter at stage 0, at stage 6, or not at all, with 60 sampled M-elements and 68 sampled N-elements per schedule. It checks label consistency, finite witnesses, atomic finite-pattern agreement, section restrictions, and adversarial finite transcripts. The same checker passed

1,925,316 assertions in each of Python 3.13.5 and 3.12.14. These executions are regression tests, not independent mathematical reviews or a proof of noncomputability. The infinite-model axioms, arbitrary-formula reduction and degree lower bound are proved in the text.

This manuscript was developed with AI assistance for mathematical exploration, proof checking, literature searches, computational tests and drafting. The argument has undergone a documented self-audit but has not been independently refereed or formally verified. The bounded literature search does not certify priority or exclude an earlier folklore observation.

References

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