Weighted Root Deletions and Coefficientwise Toeplitz Positivity

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Abstract

For independent indeterminates $a, x_1, \ldots, x_N, y_1, \ldots, y_N$, we prove that the sequence $b_k = ae_k(X) + \sum_i y_i e_k(X \setminus x_i)$ is coefficientwise totally nonnegative: every minor of its upper Toeplitz matrix has nonnegative integer coefficients. In particular, this holds for the coefficients of $(uD_z + v)\prod_i(1 + x_i z)$, coefficientwise in u, v, X. The proof realizes the sequence as sums of bordered principal minors of a star-shaped Gram pencil. A maximal-weight compression in a Schur module expresses the necessary Schur-complement characters as traces against orthogonal projections. An additional letter-content grading separates the independent root weights and yields explicit squared-norm coefficient certificates. This resolves the derivative-plus-constant branch of an AIM total-positivity question, not its other operator conjectures. The note is unrefereed and makes no absolute priority claim.

1 The theorem and its scope

For $X = (x_1, \ldots, x_N)$, let $e_k(X)$ denote the elementary symmetric polynomial, with $e_0 = 1$ and $e_k = 0$ for $k < 0$ or $k > N$. A sequence $(b_k)_{k \geq 0}$ is *coefficientwise totally nonnegative* if every finite minor of $(b_{j-i})_{i,j \geq 0}$ has nonnegative coefficients, where $b_k = 0$ for $k < 0$. We allow zero minors and include zero in $\mathbb{N}$.

Theorem 1.1. *Let a, $X = (x_1, \ldots, x_N)$, and $Y = (y_1, \ldots, y_N)$ be independent commuting indeterminates, with $N \geq 0$. Set*

$$b_k = ae_k(X) + \sum_{i=1}^{N} y_i e_k(X \setminus x_i) \quad (k \geq 0), \qquad b_k = 0 \quad (k < 0). \tag{1}$$

Every finite minor of $(b_{j-i})_{i,j \geq 0}$ belongs to $\mathbb{N}[a, X, Y]$.

The generating polynomial in this theorem is a weighted sum of the original product and its single-factor deletions:

$$\sum_k b_k z^k = a \prod_i (1 + x_i z) + \sum_i y_i \prod_{j \neq i} (1 + x_j z).$$

The index sets of the minors may have arbitrary gaps.

Corollary 1.2. *For independent u, v, X, the coefficient sequence of $(uD_z + v)\prod_i(1 + x_i z)$ is coefficientwise totally nonnegative in u, v, X.*

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Proof. Substitute $a = v$ and $y_i = ux_i$ in (1). The resulting sequence is $ve_k(X) + u(k+1)e_{k+1}(X)$. Polynomial substitution preserves coefficientwise nonnegativity, including the endpoints $u = 0$ and $v = 0$. $\square$

The 2023 AIM workshop on total positivity asked whether several classical real-root-preserving operations admit coefficientwise upgrades [1]. The listed operations include $D+a$, $zD+a$, Hadamard and factorial-Hadamard products, and a nonlinear log-concavity operation. Corollary 1.2 settles the first of these branches at every order. We do not claim to settle the other branches, general differential composition $f(D)g$, or arbitrary products of affine multipliers.

The separate family $(uk + v)e_k(X)$ was treated in an earlier unrefereed anonymous release [3]. We credit that precursor's rank-one Gram realization and graded coefficient extraction. The Schur-specialisation strategy itself is attributed to Richard Stanley in Sokal's slides [2]. The present extension handles the variable constant term $b_0 = a + \sum_i y_i$ by a bordered maximal-weight trace identity.

2 The bordered trace identity

We use classical Schur–Weyl duality, the semistandard-tableau description of polynomial general-linear characters, and the Littlewood–Richardson rule. The general-linear branching rule is recalled, for example, in Molev [5, Theorem 2.1]. The Gram-vector/central-projector positivity mechanism is classical; see Huber–Maassen [4, Section 2].

Let $V = \mathbb{C}e_0 \oplus W$, $\dim W = N$, with its standard inner product. For a block matrix write

$$A = \begin{pmatrix} h & b \\ c & D \end{pmatrix}, \qquad S = D - cb/h, \qquad h \neq 0. \tag{2}$$

Fix $r \geq 1$ and a partition λ with $\lambda_1 \leq r$. The empty partition is allowed. Put $\nu = (r, \lambda)$ and $m = r + |\lambda|$. Let U_σ permute tensor positions on $V^{\otimes m}$. For the irreducible $\mathfrak{S}_m$-character χ^ν of dimension f^ν, set

$$P_\nu = \frac{f^\nu}{m!} \sum_{\sigma \in \mathfrak{S}_m} \chi^\nu(\sigma^{-1}) U_\sigma. \tag{3}$$

This is the central orthogonal projection onto the ν-isotypic component. Let Q_r project onto basis words containing exactly r copies of e_0. Position permutations preserve this count, so P_ν and Q_r commute and their product is an orthogonal projection.

Lemma 2.1. *With the preceding notation,*

$$h^r s_\lambda(S) = \frac{1}{f^\nu} \operatorname{Tr}(P_\nu Q_r A^{\otimes m}). \tag{4}$$

Here $s_\lambda(S)$ is the polynomial character evaluated on S. In particular the expression on the left, initially written with h^{-1}, is polynomial in the entries of A.

Proof. Schur–Weyl duality identifies the image of P_ν with f^ν copies of $\mathbb{S}_\nu(V)$, on each of which $A^{\otimes m}$ acts by $\mathbb{S}_\nu(A)$. The maximal degree in the letter e_0 is $\nu_1 = r$. Indeed, in a semistandard tableau the smallest letter can occur only in the first row. Exactly r occurrences fill that row and leave a tableau of shape λ over the alphabet of W. Thus the degree-r component is, as a $\mathrm{GL}_1 \times \mathrm{GL}(W)$-module,

$$(\mathbb{C}e_0)^{\otimes r} \otimes \mathbb{S}_\lambda(W), \tag{5}$$

with multiplicity one. Equivalently this is the extremal summand of the classical branching rule.

Use the factorization

$$A = L\,\mathrm{diag}(h, S)U, \qquad L = \begin{pmatrix} 1 & 0 \\ c/h & I \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & b/h \\ 0 & I \end{pmatrix}.$$

The operator U only raises e_0-degree and its degree-preserving part is the identity. On the maximal degree of $\mathbb{S}_\nu(V)$ it therefore acts identically. The operator L only lowers that degree; its compression to the maximal degree is the identity. The middle factor acts on (5) by $h^r\mathbb{S}_\lambda(S)$. Taking the trace of this compression and restoring the multiplicity f^ν proves (4). Singular S follow by polynomial extension from invertible S. If $\ell(\lambda) > N$, both relevant Schur modules vanish. For the empty λ the identity reduces to $h^r = h^r$. □

3 Bordered principal minors and arbitrary index sets

For an $(N+1)$-square matrix indexed by $0, 1, \ldots, N$, define

$$c_k(A) = \sum_{\substack{I \subseteq [N] \\ |I|=k}} \det A[\{0\} \cup I, \{0\} \cup I] \quad (0 \le k \le N), \tag{6}$$

and $c_k = 0$ outside this range. Notice that $c_0 = h$, not 1. The block determinant identity gives $c_k = he_k(S)$ when $h \neq 0$.

Take increasing rows $i_1 < \cdots < i_r$ and columns $j_1 < \cdots < j_r$. If $i_p > j_p$, the last $r - p + 1$ rows vanish in the first p selected columns, so every determinant term is zero. Otherwise define conjugate partitions by

$$\alpha'_p = j_{r+1-p} - r + p, \qquad \beta'_p = i_{r+1-p} - r + p \quad (1 \le p \le r). \tag{7}$$

They are weakly decreasing and nonnegative, with $\beta \subseteq \alpha$. Reversing both index orders and transposing changes no determinant sign. Dual Jacobi–Trudi and Littlewood–Richardson then give

$$\det(c_{j_q - i_p}) = h^r s_{\alpha/\beta}(S) = \sum_\lambda c^\alpha_{\beta\lambda} h^r s_\lambda(S). \tag{8}$$

The coefficients are nonnegative integers. A nonzero LR coefficient forces $\lambda \subseteq \alpha$, by symmetry in the two lower LR indices, so $\lambda_1 \le \alpha_1 \le r$. Lemma 2.1 applies to every term of (8).

Proposition 3.1. *Let $A(t) = \sum_\gamma t^\gamma A_\gamma$ be a real symmetric matrix polynomial in finitely many variables, with each coefficient matrix A_γ positive semidefinite. The sequence* (6) *is coefficientwise totally nonnegative.*

Proof. Every coefficient matrix C of $A(t)^{\otimes m}$ is a sum of tensor products of PSD matrices, hence PSD. For the orthogonal projection $R = P_\nu Q_r$,

$$\mathrm{Tr}(RC) = \mathrm{Tr}(RCR) \ge 0.$$

No commutation of R with C is needed. Apply (4) termwise in (8). The resulting identities are polynomial in the entries of A and extend to $h = 0$. □

The proposition already proves the derivative case from $A = ae_0e_0^T + \sum_i x_i(e_0 + e_i)(e_0 + e_i)^T$. The independent weights require an additional argument: pointwise PSD alone does not imply positivity of polynomial coefficients.

4 Independent weights by orthogonal letter contents

Introduce variables $z_1, \ldots, z_N$, set $v_0 = e_0$ and $v_i = z_i e_0 + e_i$, and form

$$A(z) = a v_0 v_0^T + \sum_{i=1}^{N} x_i v_i v_i^T. \tag{9}$$

Its leading entry is $h = a + \sum_i x_i z_i^2$, its $0i$ entry is $x_i z_i$, and its trailing block is $\operatorname{diag}(X)$. A block determinant gives

$$\det A[\{0\} \cup I] = \left(a + \sum_{i \notin I} x_i z_i^2 \right) \prod_{i \in I} x_i.$$

The formula holds at zero x_i by polynomial identity. Summing gives

$$c_k = a e_k(X) + \sum_i x_i z_i^2 e_k(X \setminus x_i). \tag{10}$$

Some coefficient matrices in the z_i are indefinite, so Proposition 3.1 cannot simply be applied to all variables.

Proof of Theorem 1.1. Fix one straight term of (8) and its r, λ, ν, m. Expand $A(z)^{\otimes m}$ first in a, X. A word of source indices with content $\eta = (\eta_0, \eta_1, \ldots, \eta_N)$, $\sum_i \eta_i = m$, contributes

$$a^{\eta_0} \prod_i x_i^{\eta_i}\, w(z) w(z)^T,$$

where $w(z)$ is the tensor product of the corresponding v_i in that word order. Its contribution to (4) is

$$\frac{a^{\eta_0} \prod_i x_i^{\eta_i}}{f^\nu} \|P_\nu Q_r w(z)\|^2. \tag{11}$$

For $k = (k_1, \ldots, k_N)$, let w_k be the sum of basis tensors obtained by replacing k_i of the η_i copies of e_i by e_0. Then $w(z) = \sum_k z^k w_k$ and Q_r retains exactly the terms with $\eta_0 + |k| = r$.

Different k give different remaining-letter counts $(\eta_1 - k_1, \ldots, \eta_N - k_N)$. Each such content space is preserved by position permutations and hence by P_ν; distinct spaces are orthogonal. Consequently

$$\|P_\nu Q_r w(z)\|^2 = \sum_{\substack{0 \le k_i \le \eta_i \\ \eta_0 + |k| = r}} z^{2k} \|P_\nu w_k\|^2. \tag{12}$$

This is a coefficient identity, not an inference from evaluation at positive parameters.

Thus (11) is coefficientwise nonnegative in $a, X, w_i = z_i^2$. Substitute $w_i = y_i / x_i$ initially in a Laurent polynomial ring. Every summand becomes a nonnegative multiple of

$$a^{\eta_0} \prod_i x_i^{\eta_i - k_i} y_i^{k_i}, \qquad 0 \le k_i \le \eta_i.$$

All exponents are nonnegative, so this is an ordinary polynomial identity. Equation (10) becomes (1). Every straight term of (8) is therefore coefficientwise nonnegative, proving the result for arbitrary minors. Integrality follows from the original determinant in the integer polynomial ring. The zero-order minor is 1, and for $N = 0$ the matrix is aI. $\square$

This proof does not use the false principle that division of two positive-coefficient polynomials preserves coefficient signs: $(x^3 + y^3)/(x + y) = x^2 - xy + y^2$. The exponent bound above is essential.

Corollary 4.1 (Coefficient certificate). *Let $w_{\eta,k}$ refer to one fixed ordered source word of content η. In the straight contribution $h^r s_\lambda(S)$ after the weight substitution, the coefficient of $a^{\eta_0} X^{\eta_{>0}-k} Y^k$ is*

$$\frac{m!}{f^\nu \prod_{i=0}^N \eta_i!} \|P_\nu w_{\eta,k}\|^2 \quad \text{if } \eta_0 + |k| = r, \tag{13}$$

and zero otherwise.

Proof. There are $m!/\prod_i \eta_i!$ orders of the source word. They have equal projected norms, because position permutations are orthogonal and commute with P_ν. Sum (12) over them. □

5 Precedents, limits, and reproducible checks

The rank-one determinant identity, Schur–Weyl duality, general-linear branching, LR positivity, and Gram/projector method are classical. The earlier affine release [3] supplies a close methodological precursor, not a theorem we claim as our own. The precise independent-weight statement of Theorem 1.1 was not found in our bounded search through 28 September 2026. This does not establish historical priority: the bordered identity is a short application of classical branching and may have an older formulation or consequence.

The unrestricted Hadamard claim in [7] was withdrawn. The later result [6] covers a ribbon-like class of skew shapes. Neither is a proof dependency here, nor do we infer their general conjectures from Theorem 1.1. In particular, an iteration of $D + a$ cannot be justified merely by applying our theorem again to its output: that output has not been factored with coefficientwise-positive root parameters in the original variables.

The accompanying standard-library Python verifier computes 4,257 symbolic Toeplitz minors: for the derivative specialization, $N = 0, \ldots, 4$; for independent weights, $N = 0, \ldots, 3$; in both cases orders $1, \ldots, 4$, rows chosen from $0, \ldots, 4$ and containing 0, and columns from $0, \ldots, N + 4$. It compares each polynomial with an exact numerical determinant at signed integer inputs. It also checks character orthogonality through degree seven, 46 dense rational instances of (4), star principal minors, structural zeros, and negative controls.

A second verifier constructs the projected vectors in (13) and compares 34 entire symbolic polynomials with direct determinants, covering 146 nonzero coefficients. Its domain is $N = 1, 2$, $r = 1, 2, 3$, $|\lambda| \leq 3$ and $\lambda_1 \leq r$. The two scripts perform respectively 9,629 and 1,736 exact assertions, all passing under Python 3.13.5 and 3.12.14. These are finite regression checks and same-code portability tests, not a proof of the unbounded theorem, a formalization, or an independent review.

AI assistance was used in deriving, drafting, auditing, and checking this argument. This manuscript is unrefereed; no outside mathematical endorsement or exhaustive novelty certification is implied.

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