

Exact Vanishing Orders for Exponentially Deformed Toeplitz Minors

Alper Ferudun*

23 October 2026

Abstract

Let $H(z) = \prod_{j=1}^q (1 + \beta_j z) / \prod_{i=1}^p (1 - \alpha_i z)$, with positive parameters, and let T_t be the lower triangular Toeplitz matrix generated by $e^{tz} H(z)$. For increasing index tuples $C \leq R$, we give an explicit formula for the order at which $\det T_t[R, C]$ becomes nonzero:

$$\sum_{i=1}^{k-p} \max\{0, r_i - c_{i+p} - q + p\}.$$

The sum is empty when $p \geq k$. Every coefficient from that order through degree $\sum_i (r_i - c_i)$ is strictly positive. A unique maximal intermediate index tuple gives the leading coefficient as a product of two positive determinants. The proof uses elementary factor supports, interlacing and Cauchy–Binet. The classical zero-pattern classification and qualitative exponential strictification are explicitly credited; the result is presented as a quantitative refinement.

1 Statement and prior work

For a formal series $F(z) = \sum_{n \geq 0} f_n z^n$, write

$$T(F)_{r,c} = f_{r-c} \quad (r, c \geq 0), \quad f_n = 0 \quad (n < 0).$$

All index tuples have distinct, increasingly ordered, nonnegative integer entries. For a tuple $R = (r_1, \dots, r_k)$, let $|R| = \sum_i r_i$; comparison of tuples is componentwise. A lower triangular minor is structurally admissible precisely when $C \leq R$.

Fix nonnegative integers p, q , and positive real numbers $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$. Repetitions are allowed and counted separately. Put

$$H(z) = \frac{\prod_{j=1}^q (1 + \beta_j z)}{\prod_{i=1}^p (1 - \alpha_i z)}, \quad \Delta_{R,C}(t) = \det T(e^{tz} H(z))[R, C]. \quad (1)$$

Empty products equal one. We use the convention $1/a! = 0$ when a is a negative integer.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>. Unrefereed preprint, version 1.0. Licensed under CC BY 4.0.

Theorem 1. Assume $C \leq R$, with both tuples of length k . Define

$$d = |R| - |C|,$$

$$s_i^* = \begin{cases} \min\{r_i, c_{i+p} + q - p\}, & i \leq k - p, \\ r_i, & i > k - p, \end{cases} \quad (2)$$

$$\nu = |R| - |S^*| = \sum_{i=1}^{k-p} \max\{0, r_i - c_{i+p} - q + p\}. \quad (3)$$

Then $S^* = (s_1^*, \dots, s_k^*)$ is strictly increasing, $C \leq S^* \leq R$, and

$$\Delta_{R,C}(t) = \sum_{j=\nu}^d A_j t^j, \quad A_j > 0 \quad (\nu \leq j \leq d). \quad (4)$$

In particular, its order of vanishing at zero is exactly ν . Its endpoint coefficients are

$$A_\nu = \det T(H)[S^*, C] \det \left(\frac{1}{(r_i - s_j^*)!} \right)_{i,j=1}^k, \quad (5)$$

$$A_d = \det \left(\frac{1}{(r_i - c_j)!} \right)_{i,j=1}^k. \quad (6)$$

If $C \not\leq R$, the minor is identically zero.

The classical Edrei form parameterizes normalized semi-infinite totally nonnegative Toeplitz matrices by a nonnegative exponential parameter and positive pole and numerator parameters; see Rietsch [4, Theorem 1.1]. The finite-parameter zero pattern used below is part of that classical theory. Supersymmetric skew-Schur functions provide another established framework for it; see Goulden and Greene [2, Eq. (1)]. We give a direct factor-chain derivation to keep the indexing and all degenerate cases explicit.

Exponential strictification is also classical. Galashin, Karp and Lam [3, Lemma 4.6(i)], crediting Lusztig, use a positive matrix exponential and Cauchy–Binet in the upper triangular setting. Transposition gives the same qualitative mechanism relevant here. Theorem 1 quantifies the initial order and the entire coefficient support for the rational family (1); it does not claim to introduce that mechanism.

The AIM total-positivity problem list asks about vanishing nontrivial minors of semi-infinite TN Toeplitz matrices [1]. The old parameter classification is not claimed as a newly solved open problem here. A bounded primary-source review did not locate (3) together with (4)–(5) in this form; this is not certified priority.

2 Elementary factor supports

Let $U_{r,c} = 1$ when $r = c + 1$, and zero otherwise. For $a, b > 0$, set

$$B_b = I + bU, \quad G_a = (I - aU)^{-1}.$$

All matrix products are well-defined entrywise. Any Cauchy–Binet expansion of a fixed minor can be performed inside a sufficiently large finite leading square block.

Lemma 2. For increasing tuples R, C of length k ,

$$\det B_b[R, C] = \begin{cases} b^{|R|-|C|}, & c_i \leq r_i \leq c_i + 1 \text{ for every } i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\det G_a[R, C] = \begin{cases} a^{|R|-|C|}, & c_i \leq r_i < c_{i+1} \ (i < k), \quad c_k \leq r_k, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently both factors are totally nonnegative.

Proof. For the bidiagonal factor, a nonzero permutation term cannot have an inversion. Indeed, if $i < \ell$ are matched to columns $j > h$, respectively, then

$$r_i \geq c_j \geq c_h + 1 \geq r_\ell,$$

contrary to the increasing order of the rows. Thus only the identity term can survive, giving the first formula.

For the geometric factor, remove a^{r_i} from row i and a^{-c_j} from column j . Each remaining row is a prefix of ones followed by zeros, with prefix length $\ell_i = \#\{j : c_j \leq r_i\}$. The lengths are nondecreasing integers in $\{0, \dots, k\}$. The determinant is nonzero precisely when $\ell_i = i$ for every i : otherwise a row is zero or two rows coincide. In the nonzero case the determinant is one. These conditions are exactly the stated interlacing inequalities. $\square$

Lemma 3. A product of q positive bidiagonal factors has a positive $[M, C]$ minor if and only if

$$c_i \leq m_i \leq c_i + q \quad (1 \leq i \leq k).$$

A product of p positive geometric factors has a positive $[R, M]$ minor if and only if

$$M \leq R, \quad r_i \leq m_{i+p} - p \quad (1 \leq i \leq k - p). \quad (7)$$

For $p = 0$, these conditions mean $R = M$.

Proof. For bidiagonal factors, necessity follows by adding at most one to each coordinate at each stage. For sufficiency the tuples

$$X_i^{(u)} = \min\{m_i, c_i + u\}, \quad 0 \leq u \leq q,$$

are increasing and meet every one-factor condition in Lemma 2. They give a positive Cauchy–Binet summand.

For geometric factors, iteration of the strict adjacent inequalities proves necessity of (7). For sufficiency induct on p . When $p \geq 1$, choose

$$y_i = \begin{cases} \min\{r_i, m_{i+p-1} - (p-1)\}, & i + p - 1 \leq k, \\ r_i, & i + p - 1 > k. \end{cases}$$

Both sequences under the minimum are strictly increasing. The transition from capped to uncapped coordinates is also strict, since $y_i \leq r_i < r_{i+1}$. Moreover, $m_{i+p-1} \geq m_i + p - 1$, so $M \leq Y \leq R$. The tuple Y meets the criterion for $p - 1$ geometric factors.

It remains to check the final step Y to R . We have $y_i \leq r_i$. For $i < k$, both candidate values for y_{i+1} exceed r_i : one is r_{i+1} , and, when the other is present, (7) gives

$$r_i \leq m_{i+p} - p < m_{i+p} - (p-1).$$

Thus $r_i < y_{i+1}$, and the final geometric factor has positive minor. This completes the induction. $\square$

Proposition 4 (Classical zero criterion). *For H in (1),*

$$\det T(H)[R, C] > 0 \iff C \leq R \text{ and } r_i \leq c_{i+p} + q - p \quad (1 \leq i \leq k - p). \quad (8)$$

All other minors are zero.

Proof. The matrix $T(H)$ is the commuting product of the q bidiagonal factors and p geometric factors. Cauchy–Binet and Lemma 3 give necessity. For sufficiency use the intermediate tuple $m_i = \min\{r_i, c_i + q\}$. It is increasing and satisfies $C \leq M \leq R$. Also, for $i \leq k - p$,

$$m_{i+p} = \min\{r_{i+p}, c_{i+p} + q\} \geq r_i + p,$$

since each term in this minimum has the required lower bound. Hence (7) holds. The resulting factor chain has positive weight. This also treats $p = 0$ and $q = 0$. $\square$

3 The exponential factor

Write $E_t = T(e^{tz})$, so its entries below the diagonal are $t^{r-c}/(r-c)!$.

Lemma 5. *If $C \leq R$ and $d = |R| - |C|$, then*

$$\det E_t[R, C] = t^d F(R, C), \quad F(R, C) = \det \left(\frac{1}{(r_i - c_j)!} \right) > 0.$$

More precisely, $F(R, C) = f(R, C)/d!$, where $f(R, C)$ counts the legal words of d serial unit moves taking the increasing tuple C to R , without collisions.

Proof. Homogeneity of every nonzero permutation term gives the displayed power t^d . For $t > 0$, approximate E_t by the product $(I + tU/N)^N$. A legal serial word exists: move the rightmost coordinate to its destination, then the next coordinate to the left, and continue. Increasing destinations ensure that no collision occurs.

For $N \geq d$, insert any such word into d distinct stages among the N factors, using identity moves at the other stages. These are $f(R, C) \binom{N}{d}$ distinct Cauchy–Binet terms, each of weight $(t/N)^d$. Every other nonzero factor-chain term has the same total number d of unit moves but uses $h < d$ nonempty stages. For each such h , at most $\binom{N}{h} (2^k)^h$ stage patterns are possible. Their total weight therefore tends to zero as $N \rightarrow \infty$.

The coefficients of $(1 + tz/N)^N$ converge to those of e^{tz} , and the selected determinant is a finite polynomial in those coefficients. Taking the limit gives

$$\det E_t[R, C] = \frac{f(R, C)}{d!} t^d.$$

Here $f(R, C) \geq 1$. The case $d = 0$ has one empty word and determinant one. $\square$

4 Proof of the quantitative theorem

Proof of Theorem 1. If $r_i < c_i$ for some i , the first i selected rows of any lower triangular matrix can be matched nontrivially to at most $i - 1$ selected columns. Every permutation term is consequently zero.

Assume $C \leq R$. Using $T(e^{tz}H) = E_t T(H)$, Cauchy–Binet and Lemma 5 give

$$\Delta_{R,C}(t) = \sum_{C \leq S \leq R} t^{|R|-|S|} F(R, S) \det T(H)[S, C]. \quad (9)$$

The sum is over increasing tuples and is finite. By Proposition 4, a summand has positive coefficient precisely when

$$C \leq S \leq S^*.$$

Indeed, the extra restrictions are $s_i \leq c_{i+p} + q - p$ for $i \leq k - p$. The tuple S^* in (2) is itself increasing, as the minimum of increasing integer sequences, with a strict transition to its uncapped tail. Also $c_{i+p} - p \geq c_i$, giving $C \leq S^* \leq R$.

The unique feasible tuple of maximum sum is S^* . It gives the smallest power $\nu = |R| - |S^*|$ and the leading coefficient (5). The unique tuple of minimum sum is C . Since $\det T(H)[C, C] = 1$, it gives degree d and coefficient (6).

Finally, take a legal serial unit-move path from C to S^* , constructed as in Lemma 5. It stays componentwise between those endpoints, and its coordinate sums take every integer value from $|C|$ to $|S^*|$. Thus (9) has a positive term for every power from ν through d . All other terms are nonnegative, so cancellation is impossible. This proves (4). $\square$

5 Consequences, examples and scope

Corollary 6. *Replace $H(z)$ by $az^m H(z)$, with $a > 0$ and $m \geq 0$. If some $r_i < c_i + m$, the deformed minor is identically zero. Otherwise apply Theorem 1 to $R - m$ and C , and multiply every coefficient by a^k .*

Proof. The entries are $a[z^{r-c-m}]e^{tz}H(z)$. The same lower triangular matching obstruction proves the first assertion. In the other case all shifted row indices are nonnegative, and factoring a from each row gives the result. $\square$

The initial order in (3) can be computed in $O(k)$ arithmetic operations once p, q, R, C are known. It depends on the counts of positive atoms, including multiplicity, but not on their values. The initial coefficient can then be evaluated from the two determinants in (5).

For example, take $H(z) = (1 - \alpha z)^{-1}$, $R = (3, 4)$, and $C = (0, 1)$. Then $S^* = (0, 4)$, $\nu = 3$ and $d = 6$. Direct expansion gives

$$\Delta_{R,C}(t) = \frac{\alpha^3}{6}t^3 + \frac{\alpha^2}{8}t^4 + \frac{\alpha}{24}t^5 + \frac{1}{144}t^6.$$

All four coefficients in the predicted interval are positive. When $p = q = 0$, $S^* = C$ and $\nu = d$; the theorem reduces to the single homogeneous exponential minor. When $p \geq k$, $S^* = R$ and $\nu = 0$, recovering positivity at $t = 0$.

For orientation, the classical full Edrei classification also allows a parameter $\gamma > 0$ or infinitely many positive atoms. A positive exponential makes every admissible minor positive by retaining the intermediate tuple C in Cauchy–Binet. Infinitely many pole atoms allow selection of k geometric factors; infinitely many numerator atoms allow selection of at least $\max_i(r_i - c_i)$ bidiagonal factors. In either case Proposition 4 provides a positive minor, and the residual TN factor has diagonal one. The exponential-order formula in this note concerns the finite-atom rational stratum at $\gamma = 0$.

The infinite TN hypothesis cannot be replaced silently by positivity of one finite truncation. For instance, the coefficients $1, 1, \frac{1}{2}, 0$ give a TN 4×4 lower triangular block, but their zero extension has

$$\det T[R, C] = -\frac{1}{4}, \quad R = (1, 2, 3, 4), \quad C = (0, 1, 2, 3).$$

This is a hypothesis check, not a counterexample to the semi-infinite classification.

The accompanying standard-library Python checker uses exact integers and rational numbers. It verifies 143,103 scalar-minor cases, 22,904 constructive interlacing witnesses, 900 full polynomial expansions, 80 full Cauchy–Binet comparisons, and 900 independent serial-word counts. It also checks indexing and finite-truncation negative controls. Normal and optimized Python executions agree byte for byte. These computations are diagnostics; the general proof is given above.

AI assistance was used in developing the argument, checking algebra, reviewing sources, and preparing the manuscript and reproducibility files. This note is unrefereed, is not formally verified by a proof assistant, and makes no absolute-priority claim.

References

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