

Regular Totally Nonnegative Loops Without Entire Matrix Logarithms

Alper Ferudun*

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Abstract

For positive a, b, c, d , let $Q_{a,b}(t) = \begin{pmatrix} 0 & a \\ bt & 0 \end{pmatrix}$. We prove that $A(t) = \exp Q_{a,b}(t) \exp Q_{c,d}(t)$ has a regular, totally nonnegative periodic unfolding, but admits an entire complex matrix logarithm if and only if $ad = bc$. The obstruction comes from nonscalar Jordan values with eigenvalues $+1$ and -1 , which force incompatible parities for the trace of any logarithm. An explicit example has rational Taylor coefficients. The examples contradict the single-logarithm assertion of Lemma 8.4 in arXiv:0812.0840v3 and show that the known two-exponential bound is sharp even among regular totally nonnegative foldings. They do not settle the general block-Toeplitz AESW-type factorization problem. No priority claim is made.

1 Statement and conventions

For a matrix series $A(t) = \sum_{k \geq 0} A_k t^k$, its periodic *upper unfolding* X is indexed by integers and defined by

$$X_{2r+i, 2s+j} = (A_{s-r})_{ij}, \quad r, s \in \mathbb{Z}, \quad i, j \in \{1, 2\}, \quad (1)$$

where $A_k = 0$ for $k < 0$. Total nonnegativity (TN) means nonnegativity of every finite scalar minor of X , not merely nonnegativity of the entries of A_k or pointwise positivity of $A(t)$.

We use the terminology of Lam–Pylyavskyy [2]. A folding is *doubly entire* when both A and A^{-1} have entire entries. For an infinitely supported upper unitriangular TN unfolding set

$$\epsilon_i(X) = \lim_{j \rightarrow +\infty} \frac{X_{i,j}}{X_{i+1,j}}, \quad \mu_i(X) = \lim_{j \rightarrow -\infty} \frac{X_{j,i+1}}{X_{j,i}}. \quad (2)$$

Such an unfolding is *regular* if it is doubly entire and all these limits are zero. Only $i = 1, 2$ need be checked by periodicity. All denominators used below are positive in the relevant tails.

Put

$$Q_{a,b}(t) = \begin{pmatrix} 0 & a \\ bt & 0 \end{pmatrix}, \quad E_{a,b}(t) = \exp Q_{a,b}(t). \quad (3)$$

Here and throughout, an entire matrix logarithm of A means an entire map $B : \mathbb{C} \rightarrow M_2(\mathbb{C})$ with $\exp B(t) = A(t)$ for every t . No reality or branch restriction is imposed on B .

Theorem 1.1. *For every $a, b, c, d > 0$, the folding*

$$A(t) = E_{a,b}(t)E_{c,d}(t)$$

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

has determinant one and an infinitely supported, upper unitriangular, regular TN unfolding. It admits an entire matrix logarithm if and only if $ad = bc$, equivalently if and only if the two generators $Q_{a,b}(t)$ and $Q_{c,d}(t)$ commute identically.

The distinction from the source problem is important. AIM totalpos Problem 1.2 [1] asks for an AESW-type factorization of a 2 by 2 block-Toeplitz TN series with invertible constant block. It does not prescribe a single-exponential remainder. Our theorem excludes that particular simplification but does not exclude an ordered-product factorization or classify all regular remainders.

2 Positivity and regularity

Lemma 2.1. *For $a, b > 0$, the unfolding of $E_{a,b}$ is TN, upper unitriangular, and positive at every position strictly above the diagonal. The same properties hold for $E_{a,b}E_{c,d}$.*

Proof. The unfolding of $I + Q_{a,b}/m$ is upper bidiagonal, with alternating nonnegative first-superdiagonal entries $a/m, b/m$. It is TN. Its m th power is TN by Cauchy–Binet. On any fixed finite upper-triangular interval these powers converge to the exponential of the corresponding weighted shift. Products and minors on that interval use no indices outside it. Consequently each fixed minor of the unfolding of $E_{a,b}$ is a limit of nonnegative minors.

The entry at distance $j - i \geq 0$ of the exponential unfolding is the product of the intervening alternating positive weights divided by $(j - i)!$. It is positive. Products of two such upper-triangular TN matrices are TN: every relevant Cauchy–Binet sum is finite. Their diagonal entries are one, and strict upper entries remain positive. $\square$

Both Q matrices have trace zero, so $\det A = 1$. Their exponentials and inverse exponentials have entire entries. Hence $A^{-1} = E_{c,d}^{-1}E_{a,b}^{-1}$ is entire. The constant block is

$$A(0) = \begin{pmatrix} 1 & a + c \\ 0 & 1 \end{pmatrix}. \quad (4)$$

It remains to check regularity; no closure theorem for regular matrices is assumed.

Set $u = \sqrt{ab}$, $v = \sqrt{cd}$, $L = u + v$, $D = u - v$, and $\alpha = ad/(uv)$, $\beta = bc/(uv) = \alpha^{-1}$. The identity $Q_{a,b}^2 = abtI$ and the addition formulas give

$$\begin{aligned} A_{11}(t) &= \frac{1}{2}((1 + \alpha) \cosh(L\sqrt{t}) + (1 - \alpha) \cosh(D\sqrt{t})), \\ A_{22}(t) &= \frac{1}{2}((1 + \beta) \cosh(L\sqrt{t}) + (1 - \beta) \cosh(D\sqrt{t})), \\ A_{12}(t) &= \frac{(a/u + c/v) \sinh(L\sqrt{t}) + (a/u - c/v) \sinh(D\sqrt{t})}{2\sqrt{t}}, \\ A_{21}(t) &= \frac{\sqrt{t}}{2}((b/u + d/v) \sinh(L\sqrt{t}) + (b/u - d/v) \sinh(D\sqrt{t})). \end{aligned} \quad (5)$$

These are entire power series; the square roots are notation for the displayed even combinations, not a branch choice.

Write p_k, q_k, r_k, s_k for the coefficients of $A_{11}, A_{12}, A_{21}, A_{22}$, respectively. Since $|D| < L$, there are positive constants P, Q, R, S such that

$$\begin{aligned} p_k &\sim P \frac{L^{2k}}{(2k)!}, & s_k &\sim S \frac{L^{2k}}{(2k)!}, \\ q_k &\sim Q \frac{L^{2k+1}}{(2k+1)!}, & r_k &\sim R \frac{L^{2k-1}}{(2k-1)!}. \end{aligned}$$

The two parity subsequences in each limit of (2) are

$$\begin{aligned} \epsilon_1 &: \frac{p_k}{r_k}, \frac{q_k}{s_k}, & \epsilon_2 &: \frac{r_k}{p_{k-1}}, \frac{s_k}{q_{k-1}}, \\ \mu_1 &: \frac{q_k}{p_k}, \frac{s_k}{r_k}, & \mu_2 &: \frac{p_{k+1}}{q_k}, \frac{r_{k+1}}{s_k}. \end{aligned}$$

Every ratio is $O(k^{-1})$. Thus all the limits vanish. Together with Lemma 2.1 and the entire inverse, this proves the regularity assertion in Theorem 1.1.

3 The logarithm obstruction

Lemma 3.1 (Jordan parity). *Let H be a holomorphic map from a connected complex domain to $\mathrm{SL}_2(\mathbb{C})$. If H takes both a nonscalar value $I+N$ and a nonscalar value $-I+M$, where $N^2 = M^2 = 0$, then H has no holomorphic matrix logarithm on that domain.*

Proof. Suppose $H = \exp B$. Since $\exp(\mathrm{tr} B) = 1$, connectedness implies that $\mathrm{tr} B = 2\pi i k$ for a fixed integer k . Also B commutes with its exponential. The centralizer of a nonscalar 2 by 2 Jordan block consists of matrices $xI + yN$ in a suitable Jordan basis. At the value $I + N$, the equation $\exp B = I + N$ therefore forces $\exp x = 1$, whence $\mathrm{tr} B = 2x \in 4\pi i \mathbb{Z}$. At the value $-I + M$ the same argument forces $\exp x = -1$, whence $\mathrm{tr} B \in 2\pi i(2\mathbb{Z} + 1)$. These parities are incompatible. $\square$

Completion of the proof of Theorem 1.1. If $ad = bc$, then $Q_{c,d} = (c/a)Q_{a,b}$, so $Q_{a,b} + Q_{c,d}$ is an entire logarithm of A .

Suppose instead that $ad \neq bc$. At $t = -x^2$, multiplication gives

$$\mathrm{tr} A(-x^2) = 2 \cos(ux) \cos(vx) - \kappa \sin(ux) \sin(vx), \quad \kappa = \frac{ad + bc}{uv} > 2. \quad (6)$$

The strict inequality is the arithmetic–geometric mean inequality, since $(ad)(bc) = u^2 v^2$ and $ad \neq bc$. For $x_0 = \pi/(u + v)$ both sine factors are positive and

$$\mathrm{tr} A(-x_0^2) = -2 - (\kappa - 2) \sin(ux_0) \sin(vx_0) < -2.$$

Since the trace is 2 at $x = 0$, some $x_* \in (0, x_0)$ has $\mathrm{tr} A(-x_*^2) = -2$. Both $\sin(ux_*)$ and $\sin(vx_*)$ are strictly positive.

At this point A cannot equal $-I$. Otherwise $E_{c,d} = -E_{a,b}^{-1}$, and comparison of the two off-diagonal entries would give

$$\frac{c \sin(vx_*)}{v} = \frac{a \sin(ux_*)}{u}, \quad \frac{d \sin(vx_*)}{v} = \frac{b \sin(ux_*)}{u}.$$

Their quotient implies $c/d = a/b$, contrary to $ad \neq bc$. Because $\det A = 1$, Cayley–Hamilton now yields $(A(-x_*^2) + I)^2 = 0$ with $A(-x_*^2) + I \neq 0$. Equation (4) is nonscalar positive unipotent. Lemma 3.1 excludes an entire logarithm. $\square$

4 An explicit family and the scope of the correction

Take $a = b = 1$, $c = r$, $d = r^{-1}$, with $r > 0$. If $C(t) = \cosh \sqrt{t}$ and $S(t) = \sinh \sqrt{t}/\sqrt{t}$, then

$$A_r(t) = \begin{pmatrix} C(t)^2 + (t/r)S(t)^2 & (1+r)C(t)S(t) \\ (1+r^{-1})tC(t)S(t) & C(t)^2 + r t S(t)^2 \end{pmatrix}. \quad (7)$$

Theorem 1.1 says that A_r has an entire logarithm exactly when $r = 1$. For rational $r > 0$ all its Taylor coefficients are rational. More explicitly, for $k \geq 1$,

$$p_k = (1 + r^{-1}) \frac{2^{2k-1}}{(2k)!}, \quad s_k = (1 + r) \frac{2^{2k-1}}{(2k)!}, \quad r_k = (1 + r^{-1}) \frac{2^{2k-2}}{(2k-1)!},$$

and $q_k = (1 + r)2^{2k}/(2k+1)!$ for $k \geq 0$.

For $r = 4$, put $x_* = \arcsin(4/5)$ and $t_* = -x_*^2$. Then $\cos x_* = 3/5$ and

$$A_4(0) = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \quad A_4(t_*) = \begin{pmatrix} 1/5 & 12/(5x_*) \\ -3x_*/5 & -11/5 \end{pmatrix}.$$

Conjugating the latter by $\text{diag}(1, x_*)$ gives the rational matrix $\begin{pmatrix} 1/5 & 12/5 \\ -3/5 & -11/5 \end{pmatrix}$, whose trace is -2 , determinant is 1 , and sum with I is nonzero with square zero.

Corollary 4.1. *The universal bound of two entire complex matrix exponentials for holomorphic $\text{SL}_2(\mathbb{C})$ maps on $\mathbb{C}$ is sharp even when restricted to regular TN foldings.*

Proof. The upper bound is the theorem of Kutzschebauch–Studer [3], applied to the open Riemann surface $\mathbb{C}$. The folding A_4 rules out a universal bound of one. For this particular family the two positive-flow factors are already explicit. $\square$

The upper bound in Corollary 4.1 is credited prior work; it does not assert that its factors are TN. General failure of a holomorphic matrix logarithm is also known, as discussed in [3]. Our purpose is the positive two-phase classification and its direct regularity check, without an absolute-priority assertion.

The assertion contradicted here is specifically Lemma 8.4 on page 34 of arXiv:0812.0840v3 [2]. Its proposed proof differentiates an exponential of a matrix antiderivative as if the exponent commuted with its derivative. That rule is unavailable in general. Our example contradicts the assertion itself, not only that proof. The corresponding passage in the journal version has not been verified here; no claim about that version is made. Nor do these examples refute the separate curl–core–whirl factorization theorem.

To match the original lower-block convention in [1], set $F(t) = A_r(t)^T$. Its lower block-Toeplitz matrix is TN by transposition and restriction of (1); its constant block is invertible. A logarithm of F would transpose to a logarithm of A_r . Thus the examples genuinely belong to the source class, although they do not answer its full factorization question.

5 A failed truncation and reproducibility

One cannot replace a general TN series by its block-Taylor truncations without an additional positivity argument. For example, the constant-plus-linear truncation of $E_{1,1}(t)$ is

$$T(t) = \begin{pmatrix} 1 + t/2 & 1 + t/6 \\ t & 1 + t/2 \end{pmatrix}.$$

Rows 1, 2, 3, 4 and columns 2, 3, 4, 5 of its upper unfolding give

$$\det \begin{pmatrix} 1 & 1/2 & 1/6 & 0 \\ 1 & 1 & 1/2 & 0 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \end{pmatrix} = -\frac{1}{12}.$$

In contrast, the positive-bidiagonal approximants in Lemma 2.1 do preserve TN. No density theorem for arbitrary regular cores follows from this construction.

The accompanying standard-library Python checker uses exact rational arithmetic for 58,624 finite assertions, including coefficients, minors, limit-ratio identities and Jordan witnesses. It passes on two Python runtimes. A separate Node implementation uses scalar path convolution and Leibniz determinants for 951 assertions, including all 923 nonempty minors of a 6 by 6 window. The checks overlap and are not independent proofs of the analytic theorem; the all-minor and global claims rest on the written arguments above.

This unrefereed note was developed with AI assistance for literature search, derivations, exact-regression code and exposition. Its arguments have been self-audited, not independently refereed or verified in a proof assistant. The author is responsible for the claims and their stated scope.

References

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