

Paneitz Resonances and Index Growth on a Sphere Times a Hyperbolic Surface

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Abstract

Let (Σ, h) be a fixed closed connected hyperbolic surface of curvature -1 . For the product metrics $g_t = g_{S^2} \oplus t^{-1}h$, $t > 0$, with the sphere of curvature 1, we determine the full Paneitz spectrum and give exact formulas for its kernel and negative index. The scales with nonconstant null functions form a locally finite set, and explicit resonances have unbounded nullity. If ν ranges over the distinct eigenvalues of Δ_h with multiplicities m_ν , then

$$\text{ind.}(P_{g_t}) = 2t \sum_{\nu < 2/9} m_\nu \sqrt{4/9 - 2\nu} + O_h(\sqrt{t}).$$

An eigenvalue exactly at $2/9$ contributes only to the remainder, in explicit narrow scale intervals. These results give an all-scale answer to the critical-kernel question in this product family, not a classification on arbitrary conformal manifolds. The equal-scale Singer–Eastwood example and the established conformal-index theory are explicitly credited. No absolute priority claim is made.

1 The kernel question and the product family

Problem 28 in the AIM workshop compilation [1, p. 27] asks when the equation $Pf = 0$ has only constant solutions, where P is the critical conformally covariant operator. In dimension four this is the Paneitz operator. The source already notes counterexamples, so the question is not a conjecture that every kernel consists of constants. The UnsolvedMath v1.6.0 record AIM-GEOMETRY-0388 [4] also includes text from the following determinant-formula section; that separate question is not considered here.

Fix a closed connected hyperbolic surface (Σ, h) with Gaussian curvature -1 . Let g_{S^2} be the round metric of curvature 1 and set

$$M = S^2 \times \Sigma, \quad g_t = g_{S^2} \oplus t^{-1}h, \quad t > 0. \quad (1)$$

Throughout, the surface metric h is fixed; only its relative scale varies. All manifolds are smooth and without boundary. We use the nonnegative Laplacian $\Delta = \delta d$. In dimension four,

$$P_g = \Delta_g^2 + \delta_g((\frac{2}{3}R_g g - 2\text{Ric}_g)d). \quad (2)$$

The tensor is understood as an endomorphism of one-forms by raising an index. This is the standard Paneitz convention; in particular, $P_g 1 = 0$ and P_g is formally self-adjoint.

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The metric at $t = 1$ is the classical Singer–Eastwood example recorded by Eastwood and Slovák [2, pp. 10–11]. Their curvature normalization is $+4, -4$, equivalent by a common homothety to ours. They state that the compact product has kernel dimension exactly four. We recover this case while treating every $t > 0$. The inherited UnsolvedMath research report likewise rederives the equal-scale example and proposes the arbitrary-curvature product calculation.

Canzani, Gover, Jakobson and Ponge [3, Theorem 4.2] establish conformal invariance of the negative index. Their Theorem 4.8 gives unbounded negative spectra on products of hyperbolic factors by varying one hyperbolic metric. Here the sphere factor and a fixed surface yield an explicit scale count, its leading coefficient, and a separate threshold contribution. The methods are separation of variables and elementary spectral counting.

2 Spectrum and zero-mode resonances

Let $\mathcal{V}$ be the set of distinct eigenvalues of Δ_h . For $\nu \in \mathcal{V}$, let E_ν denote its eigenspace and $m_\nu = \dim E_\nu$. Connectedness gives $m_0 = 1$. The spherical harmonics $\mathcal{H}_\ell$ have eigenvalue $a_\ell = \ell(\ell + 1)$ and dimension $2\ell + 1$.

Theorem 1. *The tensor-product Laplace basis is a complete eigenbasis for P_{g_t} . On $\mathcal{H}_\ell \otimes E_\nu$ the eigenvalue is*

$$p_{\ell,\nu}(t) = (a_\ell + t\nu)^2 - \frac{2 + 4t}{3}a_\ell + \frac{4 + 2t}{3}t\nu. \quad (3)$$

Its multiplicity is $(2\ell + 1)m_\nu$; multiplicities are added when different pairs give the same numerical eigenvalue.

Proof. The scalar curvature is $R_{g_t} = 2(1 - t)$, and the Ricci endomorphism has eigenvalues 1 and $-t$ on the two factors. Thus the tensor in (2) has the constant eigenvalues $-(2 + 4t)/3$ and $(4 + 2t)/3$ there. The product Laplacian is $\Delta_{S^2} + t\Delta_h$, so substitution gives (3). The factor operators commute and have complete orthogonal eigenbases. The product volume form is t^{-1} times a fixed measure, which does not affect orthogonality or completeness. Self-adjoint ellipticity on the closed product identifies this diagonalization with the full discrete spectrum. $\square$

Theorem 2. *For $0 < t < 1$, P_{g_t} is nonnegative and its kernel consists exactly of constants. At $t = 1$,*

$$\ker P_{g_1} = \mathbb{R} \oplus (\mathcal{H}_1 \otimes E_0), \quad \dim \ker P_{g_1} = 4.$$

For every $t > 1$, P_{g_t} has a negative eigenvalue. The set

$$\mathcal{Z} = \{t > 0 : \dim \ker P_{g_t} > 1\}$$

is locally finite; consequently the kernel consists of constants on an open dense set of scales. For each $\ell \geq 1$,

$$t_\ell = \frac{3\ell(\ell + 1) - 2}{4} \in \mathcal{Z}, \quad \dim \ker P_{g_{t_\ell}} \geq 2\ell + 2, \quad (4)$$

and $p'_{\ell,0}(t_\ell) = -4a_\ell/3$. At every scale the exact kernel is

$$\ker P_{g_t} = \bigoplus_{p_{\ell,\nu}(t)=0} \mathcal{H}_\ell \otimes E_\nu. \quad (5)$$

Proof. For $\ell = 0$, the eigenvalue is $t\nu[t\nu + (4 + 2t)/3]$, positive when $\nu > 0$. For $\ell \geq 1$, write it as

$$a_\ell[a_\ell - (2 + 4t)/3] + t\nu[2a_\ell + t\nu + (4 + 2t)/3]. \quad (6)$$

Since $a_\ell \geq 2$, this is positive for $t < 1$. At $t = 1$ it vanishes exactly when $\ell = 1$ and $\nu = 0$. Also $p_{1,0}(t) = 8(1 - t)/3$, proving the negative-eigenvalue assertion.

Fix $[\epsilon, T] \subset (0, \infty)$ and put $w = a_\ell + t\nu$, $c = (2 + 4T)/3$. Formula (3) gives $p_{\ell,\nu}(t) \geq w^2 - cw$. A nonconstant zero mode therefore has $a_\ell \leq c$ and $\nu \leq c/\epsilon$. Only finitely many pairs qualify. For $\ell \geq 1$ the scale polynomial has positive constant term $a_\ell(a_\ell - 2/3)$, so is not identically zero. For $\ell = 0$, $\nu > 0$, it has no positive root. Thus only finitely many exceptional scales lie in this compact interval.

Finally $p_{\ell,0}(t) = a_\ell[a_\ell - (2 + 4t)/3]$ gives (4) and the transverse derivative. The kernel formula follows from Theorem 1, including any coincidences of resonances. The lower bound in (4) does not exclude additional mixed modes. $\square$

These results distinguish a trivial nonconstant kernel from positivity: for all nonexceptional scales above one the kernel consists of constants although the operator has negative eigenvalues.

3 Exact negative-index formula

Let $\text{ind.}(P)$ be the number of strictly negative eigenvalues, counted with multiplicity. Define

$$D_t(\nu) = (2t + 1)^2 - 18t(t + 1)\nu, \quad (7)$$

$$x_\pm(t, \nu) = \frac{2t + 1}{3} - t\nu \pm \frac{\sqrt{D_t(\nu)}}{3} \quad (8)$$

when $D_t(\nu) \geq 0$. The spherical counting functions are

$$F_{<}(x) = \sum_{a_\ell < x} (2\ell + 1), \quad F_{\leq}(x) = \sum_{a_\ell \leq x} (2\ell + 1),$$

with degrees $\ell \geq 0$. Both vanish for $x < 0$, while $F_{<}(0) = 0$ and $F_{\leq}(0) = 1$. For $x > 0$ put $r(x) = (\sqrt{1 + 4x} - 1)/2$. Then

$$F_{<}(x) = \lceil r(x) \rceil^2, \quad F_{\leq}(x) = (\lfloor r(x) \rfloor + 1)^2. \quad (9)$$

Theorem 3. *For every $t > 0$,*

$$\text{ind.}(P_{g_t}) = \sum_{\substack{\nu \in \mathcal{V} \\ D_t(\nu) > 0}} m_\nu [F_{<}(x_+(t, \nu)) - F_{\leq}(x_-(t, \nu))]. \quad (10)$$

This sum is finite. A zero discriminant contributes no negative eigenvalue, though it may contribute to (5).

Proof. For fixed t, ν , (3) is a monic quadratic in $a = a_\ell$ with discriminant $4D_t(\nu)/9$. Its roots are (8). It is negative precisely between the two distinct real roots. The weighted count in that open interval is the difference in (10); the distinct strict and inclusive conventions remove both zero endpoints. Moreover $D_t(\nu) > 0$ implies

$$\nu < \frac{(2t + 1)^2}{18t(t + 1)} = \frac{2}{9} + \frac{1}{18t(t + 1)}, \quad (11)$$

leaving only finitely many surface eigenvalues. Formula (9) follows from $1 + 3 + \dots + (2L + 1) = (L + 1)^2$. $\square$

4 Large-scale growth and the threshold

Theorem 4. *For fixed (Σ, h) , as $t \rightarrow \infty$,*

$$\text{ind.}(P_{g_t}) = tC(h) + O_h(\sqrt{t}), \quad C(h) = 2 \sum_{\substack{\nu \in \mathcal{V} \\ \nu < 2/9}} m_\nu \sqrt{\frac{4}{9} - 2\nu}. \quad (12)$$

In particular $C(h) \geq 4/3$. If the first positive eigenvalue of Δ_h is greater than $2/9$, then $C(h) = 4/3$. If $2/9 \in \mathcal{V}$, its exact contribution to the index is

$$m_{2/9} \sum_{\substack{\ell \geq 1 \\ (9a_\ell - 6)/4 < t < 9a_\ell/4}} (2\ell + 1), \quad (13)$$

which is $O_h(\sqrt{t})$ and contributes no linear term.

Proof. The surface has a discrete spectrum tending to infinity, hence a least eigenvalue $\nu_* > 2/9$. By (11), every $\nu \geq \nu_*$ ceases to contribute once $t(t+1) > 1/[18(\nu_* - 2/9)]$. Thus only a fixed finite set of surface eigenvalues enters the asymptotic.

For $0 \leq \nu < 2/9$, expand (8) as

$$x_\pm(t, \nu) = t\alpha_\pm(\nu) + O_\nu(1), \quad (14)$$

$$\alpha_\pm(\nu) = \frac{2}{3} - \nu \pm \sqrt{\frac{4}{9} - 2\nu}. \quad (15)$$

Their sum is positive and their product is $\nu(\nu + 2/3)$, so both coefficients are nonnegative, and positive for $\nu > 0$. For $\nu = 0$ one has exactly $x_- = 0$, $x_+ = (4t + 2)/3$. The explicit counting functions give, for $x \geq 0$,

$$F_{<}(x) = x + O(\sqrt{x} + 1), \quad F_{\leq}(x) = x + O(\sqrt{x} + 1). \quad (16)$$

Each surface eigenvalue below $2/9$ therefore contributes

$$2m_\nu t \sqrt{4/9 - 2\nu} + O_\nu(m_\nu \sqrt{t}).$$

For $\nu = 2/9$ the quantity $D_t(\nu)$ is exactly 1, with roots $x_- = 4t/9$ and $x_+ = (4t + 6)/9$. This gives (13). The root interval has width $2/3$, while consecutive spherical eigenvalues differ by at least two. At most one degree is present, and its multiplicity is $O(\sqrt{t})$. Summing the finitely many contributions proves the result. Since $m_0 = 1$, the constant surface mode supplies $4/3$ to $C(h)$. $\square$

The error constant depends on the fixed surface. No uniform estimate over hyperbolic moduli space or computation of an unspecified surface's Laplace spectrum is asserted. Formula (10) is exact in terms of that spectrum, including its multiplicities.

5 Nullity and conformal consequences

Corollary 5. *For each fixed h ,*

$$\dim \ker P_{g_t} = O_h(\sqrt{t}) \quad (t \geq 1), \quad \limsup_{t \rightarrow \infty} \frac{\dim \ker P_{g_t}}{\sqrt{t}} \geq \frac{4}{\sqrt{3}}. \quad (17)$$

There are infinitely many conformal classes on the fixed manifold $S^2 \times \Sigma$ distinguished by their negative Paneitz indices. The index and nullity statements remain true after each g_t is rescaled to unit volume.

Proof. For a nonconstant zero mode, $D_t(\nu) \geq 0$. For $t \geq 1$, the non-strict version of (11) bounds $\nu \leq 1/4$. There are finitely many such surface eigenvalues, and for each there are at most two spherical degrees. Formula (8) bounds each a_ℓ by $(4t+2)/3$, so each spherical multiplicity is $O(\sqrt{t})$. Adding these terms and the constant mode proves the upper bound. The lower bound follows from (4) and $t_\ell \sim 3\ell^2/4$.

For $\hat{g} = e^{2\omega}g$, critical covariance gives $P_{\hat{g}} = e^{-4\omega}P_g$ and $dv_{\hat{g}} = e^{4\omega}dv_g$. Both the kernel and the quadratic form $\int u P_g u dv_g$ on smooth functions are unchanged. Its maximal negative-subspace dimension is the negative index, so this index is conformally invariant, as in [3]. By (12) it is unbounded in our family, distinguishing infinitely many classes. Constant positive rescalings preserve the same quantities, including the rescaling to unit volume. $\square$

6 Scope and reproducibility

The results characterize the critical kernel throughout the product family (1). They do not characterize arbitrary closed four-manifolds, higher-dimensional critical operators, or the determinant question adjoining AIM Problem 28. The equal-scale example and the general conformal-index principle are established results. Historical priority for the explicit all-scale formulas is not certified by the bounded literature search.

The accompanying source archive includes the analytic argument and a SymPy/Fraction checker. Its 12,000 exact checks cover curvature coefficients, polynomial identities, pure-sphere resonances, strict endpoint conventions and rational weighted counts. Normal and optimized Python executions agree. The test surface-mode values are synthetic and are not asserted to be realizable hyperbolic spectra. The analytic proofs, not those finite tests, establish the unrestricted statements in this note. This preprint is unrefereed. AI assistance was used in derivation, literature search, checking and exposition; the review is self-audit, not independent peer review or formal verification.

References

- [1] American Institute of Mathematics, *Conformal Structure in Geometry, Analysis, and Physics*, workshop problem compilation (2003), Problem 28, p. 27. [Available from AIM](#).
- [2] M. Eastwood and J. Slovák, *A Primer on Q-Curvature*, workshop notes (2003), pp. 10–11. [Author-hosted notes](#).
- [3] Y. Canzani, A. R. Gover, D. Jakobson and R. Ponge, Conformal invariants from nodal sets. I. Negative eigenvalues and curvature prescription, *International Mathematics Research Notices* 2014 (2014), 2356–2400. [arXiv:1208.3040v3](#).
- [4] ulamai, *UnsolvedMath*, frozen version 1.6.0, record AIM-GEOMETRY-0388 and accompanying research report. <https://huggingface.co/datasets/ulamai/UnsolvedMath>.