

One Compact Amoeba Intersection, Several Mixed Cells

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Abstract

We give explicit obstructions to assigning one binomial system to a connected compact intersection of amoebas. For $f = w - 1$ and $g = (z - 10)(z - 12)(z + 11)(z + 13) - 525w$, the intersection is a single compact interval and the original system has four distinct real torus solutions. The canonical Ronkin spines have four transverse intersections, each dual to a fully mixed cell of volume one, while the normalized mixed Monge–Ampère mass of the compact component is four. Moreover, every binomial system from any single fully mixed cell for these Newton polytopes has at most two real torus solutions, even after coarsening. Thus a real-preserving bijection with one such system is impossible. For every $d \geq 2$, we also construct a compact component of mass d containing d unit canonical-spine cells. The constructions address the unrestricted single-cell questions in the AIM amoeba list. They do not concern a sum of cells or impose full-dimensional Newton polytopes, and they do not contradict tropical lifting or patchworking.

1 The question and the conventions

For a Laurent polynomial h in two variables, write

$$\mathcal{A}(h) = \{\text{Log}(z, w) : h(z, w) = 0, (z, w) \in (\mathbb{C}^*)^2\}, \quad \text{Log}(z, w) = (\log |z|, \log |w|).$$

Its Ronkin function is

$$N_h(x, y) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log |h(e^{x+i\theta}, e^{y+i\phi})| d\theta d\phi.$$

The *canonical Ronkin spine* $S(h)$ is the corner locus of the maximum of the affine extensions of N_h from the components of $\mathbb{R}^2 \setminus \mathcal{A}(h)$ [2, Section 2]. This construction uses Ronkin coefficients, which need not be the logarithms of the absolute values of the original coefficients.

Question A.2 in the AIM list, contributed by Frédéric Bihan, asks whether a compact connected amoeba-intersection component has mass equal to the volume of a corresponding mixed cell, and whether its solutions correspond to those of a binomial system while preserving real solutions [1]. We use the normalization

$$\mu_{f,g} = 2! \widetilde{M}(N_f, N_g).$$

Passare–Rullgård’s local root-count formula gives $\mu_{f,g}(K)$ as the number of solutions above a compact component K , counted with multiplicity [2, Theorem 5 and Corollary 1]. This is an established fact, not an extra phase-stability assumption introduced here.

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A fully mixed cell in dimension two is a Minkowski sum of one edge from each Newton polytope. Its volume is the absolute determinant of the two edge vectors. A binomial system for that cell has the two edge endpoints as the two exponents of each equation, with nonzero coefficients. The AIM wording does not specify how to choose a subdivision. Our volume statement fixes the canonical-spine subdivision; our real-solution obstruction holds for every single fully mixed cell in any lattice subdivision of the same Newton polytopes.

Theorem 1. *Let*

$$P(z) = (z - 10)(z - 12)(z + 11)(z + 13) = z^4 + 2z^3 - 265z^2 - 266z + 17160, \quad (1)$$

$$f(z, w) = w - 1, \quad g(z, w) = P(z) - 525w. \quad (2)$$

Then the following statements hold.

1. *The entire intersection $\mathcal{A}(f) \cap \mathcal{A}(g)$ is one compact connected interval $K \subset \{y = 0\}$.*
2. *The system $f = g = 0$ has four distinct real torus solutions, all above K , and $\mu_{f,g}(K) = 4$.*
3. *The spines intersect transversely at $(\log r, 0)$, for $r = 10, 11, 12, 13$. The four dual fully mixed cells have volume one. No one of them accounts for the mass or the number of complex solutions above K .*
4. *Every binomial system for any one nondegenerate fully mixed cell of $\text{New}(f), \text{New}(g)$, with arbitrary nonzero coefficients, has at most two real torus solutions. Consequently no bijection from the original solution set to a single such binomial solution set can send all real solutions to real solutions.*

The first Newton polytope is a segment. No assertion is made about a modification of the question requiring full-dimensional supports. The theorem also does not exclude a correspondence with a *union* of binomial start systems.

2 An explicit connected lemniscate projection

We prove the connectedness assertion without inferring topology from a plot. Set $u = z + \frac{1}{2}$ and $m = \frac{533}{4}$. Then

$$P(z) = \left(u^2 - \frac{441}{4}\right) \left(u^2 - \frac{625}{4}\right) = (u^2 - m)^2 - 529. \quad (3)$$

For $|t| \leq 525$, all four roots of $P(z) = t$ are

$$z_{s,e}(t) = -\frac{1}{2} + s\sqrt{m + e\sqrt{529 + t}}, \quad s, e \in \{1, -1\}. \quad (4)$$

Use the square roots with positive real part. These branches are analytic on a neighborhood of the closed disk: $529 + t$ has positive real part, and

$$|\sqrt{529 + t}| \leq \sqrt{1054} < m$$

ensures that both outer radicands have positive real part. The inner root is nonzero, so the two outer squared values differ. The outer roots are nonzero and have positive real part; hence the four branches are distinct. None of their values is zero because $P(0) = 17160 > 525$.

The four curves

$$C_{s,e} = \{z_{s,e}(525e^{i\theta}) : 0 \leq \theta \leq 2\pi\}$$

exhaust the level set $|P(z)| = 525$. Each is compact and connected in $\mathbb{C}^*$, so its radial image is a compact positive interval. Put

$$\alpha = \sqrt{m - \sqrt{1054}}, \quad \beta = \frac{\sqrt{525}}{2}, \quad \gamma = \frac{\sqrt{541}}{2}, \quad \delta = \sqrt{m + \sqrt{1054}}.$$

The values of the branches at $t = 525$ and $t = -525$ show that these four radial images contain the intervals

$$\begin{aligned} I_1 &= [\alpha - \tfrac{1}{2}, \beta - \tfrac{1}{2}], & I_2 &= [\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2}], \\ I_3 &= [\gamma - \tfrac{1}{2}, \delta - \tfrac{1}{2}], & I_4 &= [\gamma + \tfrac{1}{2}, \delta + \tfrac{1}{2}]. \end{aligned} \tag{5}$$

All endpoints are positive. Exact squaring gives

$$\alpha < \frac{41}{4}, \quad \beta > \frac{45}{4}, \quad \gamma < \frac{47}{4}, \quad \delta > \frac{51}{4}.$$

For example, the first and last inequalities reduce to $1054 > (451/16)^2$ and $1054 > (469/16)^2$, respectively. It follows that

$$\beta - \alpha > 1, \quad \gamma - \beta < \frac{1}{2} < 1, \quad \delta - \gamma > 1.$$

Thus the intervals in (5) overlap in the displayed order. The four radial images, each containing one of these intervals, have connected union. Their union is some compact interval $[r_-, r_+] \subset (0, \infty)$. Since $\mathcal{A}(f) = \{y = 0\}$ and $g(z, w) = 0$ with $|w| = 1$ is equivalent to $|P(z)| = 525$, we obtain

$$\mathcal{A}(f) \cap \mathcal{A}(g) = [\log r_-, \log r_+] \times \{0\}.$$

This describes the entire intersection; there are no omitted lemniscate branches or additional components.

Setting $w = 1$, the original four roots are

$$z = -\frac{1}{2} \pm \alpha, \quad z = -\frac{1}{2} \pm \delta.$$

They are distinct, nonzero and real. The local root-count corollary cited above gives $\mu_{f,g}(K) = 4$. Alternatively, independently rotating the coordinates in the two equations reduces them to $P(e^{ia}z) = 525e^{ib}$. Formula (4) gives four simple roots for every phase; all their logarithms belong to K . The phase-average formula therefore gives the same mass directly.

3 Computing the canonical spines

Lemma 2. *Let Q be monic of degree d , with nonzero roots having distinct moduli $0 < \rho_1 < \dots < \rho_d$, and let $0 < L < |Q(0)|$. For $h(z, w) = Q(z) - Lw$, the canonical spine is the corner locus of*

$$\max \left\{ \sum_{j=1}^d \max(x, \log \rho_j), \log L + y \right\}. \tag{6}$$

It meets $S(w - 1)$ transversely at the d points $(\log \rho_j, 0)$. Every corresponding fully mixed cell has volume one.

Proof. Jensen's formula gives

$$N_Q(x) = \sum_{j=1}^d \max(x, \log \rho_j).$$

Averaging first in the w -phase gives

$$N_h(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \max\{\log |Q(e^{x+i\theta})|, \log L + y\} d\theta.$$

For x different from every $\log \rho_j$, sufficiently negative y makes this equal to $N_Q(x)$. All $d + 1$ affine pieces of N_Q consequently occur as complement orders $(k, 0)$, $0 \leq k \leq d$. For sufficiently positive y , the affine value $\log L + y$ occurs with order $(0, 1)$. These are all lattice points of $\text{New}(h) = \text{conv}\{(0, 0), (d, 0), (0, 1)\}$. The injective order map leaves no further complement component; therefore (6) is exactly the defining maximum for the canonical spine. The face restriction formula of [2, Proposition 2] gives the same coefficients.

Finally $N_Q(x) \geq \log |Q(0)| > \log L$ for every x . On $y = 0$ the last term in (6) is inactive. Every bottom breakpoint changes the slope by one and has a vertical spine segment through it. The other spine is the horizontal line $y = 0$. The dual edge vectors are $(1, 0)$ and $(0, 1)$, of determinant one. $\square$

Applying the lemma to (1) proves the third assertion of Theorem 1. Each unit cell gives a binomial system with exactly one torus solution, whereas the compact component has four.

Remark 3. Coefficientwise tropicalization is different. In this example the bottom coefficient heights have upper hull through exponents 0, 2, 4:

$$266^2 < 17160 \cdot 265, \quad 2^2 < 265, \quad 17160 < 265^2.$$

At $y = 0$, the two breakpoints are $x = \frac{1}{2} \log(17160/265)$ and $x = \frac{1}{2} \log 265$. Their mixed cells have volume two. A further coarsening can have one cell of volume four. We do not claim that every subdivision fails to match the complex mass. The real obstruction below survives every such choice.

4 Why one binomial cannot preserve the real solutions

The first Newton polytope in (2) is the primitive vertical segment. Up to an invertible monomial factor, its binomial equation is $aw + b = 0$, with $ab \neq 0$, and fixes a unique $w \in \mathbb{C}^*$. Substituting this into the second binomial yields

$$z^k = c, \quad c \in \mathbb{C}^*, \quad k \in \mathbb{Z} \setminus \{0\}.$$

The condition $k \neq 0$ is precisely nondegeneracy of the fully mixed cell. If w is not real there are no real torus solutions. If it is real, the last equation has at most two nonzero real roots, regardless of the nonzero coefficients. In the present Newton triangle, $1 \leq |k| \leq 4$, but the real bound needs no such bound. It follows that any one binomial system has at most two real solutions. Since all four original solutions are real, even an injection that sends them to real solutions is impossible. This proves the final assertion of Theorem 1.

This obstruction is to a single binomial system. Standard polyhedral homotopy uses several start systems when there are several mixed cells. Real preservation also requires hypotheses: Ergür–de Wolff provide sufficient patchworking conditions [4], and Lee–Lindberg–Rodriguez explicitly discuss the failure of a general real-completeness guarantee outside those conditions [5, Remark 4.2]. We do not claim that this general limitation is new. The explicit compact component and the subdivision-independent single-system obstruction are the features isolated here.

5 An unbounded discrepancy

Theorem 4. *For every integer $d \geq 2$ there is a monic polynomial Q of degree d , with distinct nonzero root moduli, for which*

$$\mathcal{A}(w - 1) \cap \mathcal{A}(Q(z) - w/2)$$

is one compact connected component of normalized mixed mass d . Its canonical spines have d transverse intersections, each dual to a unit fully mixed cell.

Proof. Write $\zeta_j = e^{2\pi i j/d}$ and, for small $\varepsilon > 0$, set

$$Q_\varepsilon(z) = \prod_{j=0}^{d-1} (z - (1 + j\varepsilon)\zeta_j).$$

At $\varepsilon = 0$, this is $z^d - 1$, all of whose critical values equal -1 . Roots of the derivative and their images depend continuously on the coefficients. Hence, for sufficiently small $\varepsilon > 0$, all critical values remain outside a disk of radius strictly greater than $1/2$. The proper polynomial map above that disk is an unramified covering of degree d . A covering of a simply connected disk is trivial, so there are d holomorphic inverse branches on a neighborhood of $\{|t| \leq 1/2\}$.

At $\varepsilon = 0$, these inverse branches are $\zeta_j(1 + t)^{1/d}$. Their moduli at $t = -1/2$ and $t = 1/2$ are $2^{-1/d} < 1$ and $(3/2)^{1/d} > 1$, respectively. The implicit-function theorem and compactness allow the branches to be labeled continuously for small ε , preserving these strict endpoint inequalities. Each branch image of $|t| = 1/2$ has a radial-image interval containing 1. The union of these d intervals is therefore connected. It is compact and avoids zero, because

$$|Q_\varepsilon(0)| = \prod_{j=0}^{d-1} (1 + j\varepsilon) \geq 1 > \frac{1}{2}.$$

This union is exactly the radial projection of $|Q_\varepsilon| = 1/2$, so the amoeba intersection is a single compact interval K . Every phase equation has d simple roots above K , giving mass d . The root moduli $1 + j\varepsilon$ are distinct, and Lemma 2 gives the d unit cells. $\square$

6 Scope, related results and reproducibility

The volume comparison here uses the canonical Ronkin-spine subdivision, not an unspecified degeneration. A union of cells can carry the full mass. Juhnke-Kubitzke–de Wolff give bounds for numbers of amoeba-intersection components in terms of spine intersections and mixed cells under explicit genericity assumptions [3, Theorems 3.10–3.11]. They do not assert a component-to-single-cell bijection. We use direct formulas rather than invoking their genericity hypotheses for the segment-supported polynomial $w - 1$. The second question in AIM A.2 has the negative answer established above for a single binomial system; the unspecified subdivision in the first question requires the stated convention. No result is claimed for a strengthened version with additional support or patchworking assumptions.

The source package includes a standard-library Python checker. Its 1,104 exact rational checks verify the quartic identities, the strict overlap inequalities, eight real level roots using Sturm sequences, all fourteen nonvertical lattice edges of the Newton triangle, and auxiliary quadratic identities. These finite checks support the written proofs; they do not replace the analytic connectedness or arbitrary-degree arguments. The same output is obtained with Python optimization enabled.

This manuscript was developed with AI assistance in the derivations, literature search, exact checks and exposition. It has been self-audited and is unrefereed; it has not undergone independent review or proof-assistant verification. A bounded primary-source search did not locate this particular compact-component construction or its arbitrary-degree version. No absolute priority claim is made.

References

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