

Cyclic Vertex Groups and Integral Toric Restriction

Alper Ferudun*

10 October 2026

Abstract

Let K be the dual boundary complex of a compact simple polytope, and let an integral matrix define a coordinate-torus quotient, with nonsingular restriction at each vertex. We show that restriction from coordinate-torus equivariant cohomology to the kernel subgroup is surjective over $\mathbb{Z}$ exactly when all vertex cokernels are cyclic. Over $\mathbb{F}_p$, the criterion is instead that every vertex determinant be prime to p . For a connected kernel, these also characterize the Borel Kirwan map. The proof calculates the regular-sequence condition in the established additive Koszul model by localizing at arithmetic fibers. We include fixed-sector criteria and an effective compact three-dimensional example with primitive normals and unit facet labels whose disconnected reducing group has a surjective integral Kirwan map despite noncyclic vertex groups. All statements concern Borel cohomology; no coarse-space or integral Chen–Ruan product classification is asserted. This manuscript is unrefereed.

1 The two restriction maps

Let P be a compact simple n -polytope, $n \geq 1$, with m facets, and let K be the boundary complex of its dual polytope, on $[m]$. Let $B \in \mathbb{Z}^{n \times m}$ satisfy $\det B_\sigma \neq 0$ for every maximal face $\sigma \in K$, where B_σ is the submatrix on its n columns. For a symplectic interpretation, the columns are the labelled facet normals of P . The induced homomorphism of compact tori

$$b : T = (S^1)^m \longrightarrow R = (S^1)^n$$

is surjective. Set $G = \ker b$, possibly disconnected, and let $Z_K = (D^2, S^1)^K$ be the moment-angle complex. For $k = \mathbb{Z}$ or $\mathbb{F}_p$, put

$$A_k = k[K], \quad u_i = \sum_{j=1}^m B_{ij} x_j, \quad I_k = (u_1, \dots, u_n),$$

where x_j has cohomological degree two. We distinguish

$$\begin{aligned} \rho_k &: H_T^*(Z_K; k) \longrightarrow H_G^*(Z_K; k), \\ \kappa_k &: H_G^*(\mathbb{C}^m; k) = H^*(BG; k) \longrightarrow H_G^*(Z_K; k). \end{aligned}$$

Both are natural restriction maps. In a symplectic toric presentation, κ_k is the Borel, or Haefliger, Kirwan map. We never replace its target by the integral cohomology of the coarse quotient. Write $C_\sigma = \mathbb{Z}^n / B_\sigma \mathbb{Z}^n$; its Smith invariants agree with those of the finite vertex stabilizer.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: [AlperTheKing](https://github.com/AlperTheKing).

Theorem 1.1. *The following conditions are equivalent:*

1. $\rho_{\mathbb{Z}}$ is surjective;
2. $u_1, \dots, u_n$ is an $A_{\mathbb{Z}}$ -regular sequence;
3. $\text{rank}_{\mathbb{F}_p} B_{\sigma} \geq n - 1$ for every prime p and maximal face σ ;
4. every C_{σ} is cyclic, including the trivial group;
5. the gcd of the $(n - 1)$ -minors of each B_{σ} is one.

The empty minor is one when $n = 1$. Under these conditions the natural map gives a ring isomorphism $H_G^*(Z_K; \mathbb{Z}) = A_{\mathbb{Z}}/I_{\mathbb{Z}}$.

Theorem 1.2. *For a prime p , the map $\rho_{\mathbb{F}_p}$ is surjective if and only if $p \nmid \det B_{\sigma}$ for every maximal face σ . In that case its target is the ring $A_{\mathbb{F}_p}/I_{\mathbb{F}_p}$.*

Corollary 1.3. *If $B(\mathbb{Z}^m) = \mathbb{Z}^n$, equivalently G is connected, then both theorems remain true with ρ replaced by κ .*

These give an explicit arithmetic criterion for a specified part of Goldin’s question 10.3 in the AIM moment-map problem list [1, Section 10]. Lerman’s subsequent comment distinguishes Haefliger from Chen–Ruan cohomology. The general disconnected Kirwan problem and that choice of cohomology theory are not resolved here.

The essential topological machinery is due to Luo–Matsumura–Moore [6] and Franz [2]. Goldin–Holm–Knutson [3, Section 9.2] already prove integral surjectivity for weighted projective spaces, give a product obstruction, and explicitly discuss cyclic isotropy as a necessary condition in their unit-label setting. Holm–Matsumura [4, Section 6] also discuss the odd torsion in such products. A 2006 Holm talk, based on work with Tolman, announces isotropy hypotheses for integral surjectivity [5]; the in-preparation work is cited in [3, 4]. We do not claim priority for these observations. Our subject is the vertex calculation of the regular-sequence criterion, with the maps and connectedness hypotheses made explicit. Pomerleano–Teleman’s broader integral theorem [7], including its proper-moment-map extension, uses inversion of stabilizer orders and does not directly replace the bad-prime integral calculation below.

2 Koszul restriction and local regularity

The natural additive model in [6, Theorem 3.3] identifies $H_G^*(Z_K; \mathbb{Z})$ with the homology of the Koszul complex on the u_i with coefficients in $A_{\mathbb{Z}}$, and identifies $\rho_{\mathbb{Z}}$ with inclusion of its length-zero term. The regular-sequence equivalence is [6, Corollary 4.11 and Theorem 5.1]. For both coefficient choices we can use [2, Proposition 6.2 and (6.8)]. The quasi-isomorphism there sends a length-zero element f to f . Lemma 5.1 and Propositions 5.2 and 5.4 of that paper identify the model and its restriction map with Borel cochains. The coefficient ring may be any PID, and the subgroup need not be connected. A free action or inversion of isotropy orders is not needed for this Borel statement.

If polynomial internal degree is d and Koszul homological degree is i , the total cohomological degree is $2d - i$. Thus restriction has image $H_0 = A_k/I_k$ and is onto precisely when all positive Koszul homology vanishes. We use this model only additively. Its ordinary Koszul product need not be the cup product, especially at 2 [2]. When restriction is onto, however, its actual kernel is I_k , and the quotient-ring assertion follows from the actual cohomology map.

For clarity, Koszul acyclicity tests the stated ordered sequence here. Give the variables and exterior generators internal degree one, so the differential preserves internal degree. The mapping-cone sequence for adjoining u_n shows that vanishing of the final H_1 makes multiplication by u_n surjective on the preceding H_1 and injective on the preceding H_0 . A positive-degree element cannot act surjectively on a nonzero graded module bounded below. Inductively the earlier H_1 vanishes and the sequence is regular. Conversely, regular sequences are Koszul-acyclic by the same mapping-cone induction. Initial quotients are proper since their degree-zero part is k .

Every field face ring $k[K]$ is Cohen–Macaulay of dimension n by Reisner’s criterion [8, Theorem 5.1]: all links in a polytopal boundary have sphere homology. The monomial basis makes $A_{\mathbb{Z}}$ flat over $\mathbb{Z}$. At a prime over (p) its local map from $\mathbb{Z}_{(p)}$ has Cohen–Macaulay closed fiber; the flat local criterion [9, Tag 045J] implies that $A_{\mathbb{Z}}$ is Cohen–Macaulay there. At primes over zero use $\mathbb{Q}[K]$. In particular, at

$$\mathfrak{m}_p = (p, x_1, \dots, x_m)$$

the local dimension is $n + 1$, as is also seen on its coordinate components $\mathbb{Z}[x_j : j \in \sigma]_{(p, x_\sigma)}$.

Global regularity of positive-degree homogeneous elements is detected at these $\mathfrak{m}_p$. Indeed, a nonzero homogeneous kernel element in any initial quotient has a proper homogeneous annihilator. Its degree-zero part is a proper ideal of $\mathbb{Z}$, contained in some (p) , and its positive-degree part is contained in $(x_1, \dots, x_m)$. The element therefore survives localization at that $\mathfrak{m}_p$. The same argument over a field uses the homogeneous maximal ideal $(x_1, \dots, x_m)$.

3 The arithmetic dimension calculation

Lemma 3.1. *For every prime p ,*

$$\dim(A_{\mathbb{Z}}/I_{\mathbb{Z}})_{\mathfrak{m}_p} = \max\{1, n - \min_{\sigma} \operatorname{rank}_{\mathbb{F}_p} B_{\sigma}\}, \quad (1)$$

$$\dim(A_{\mathbb{F}_p}/I_{\mathbb{F}_p})_{(x)} = n - \min_{\sigma} \operatorname{rank}_{\mathbb{F}_p} B_{\sigma}. \quad (2)$$

The minima are over maximal faces.

Proof. The minimal coordinate primes of $A_{\mathbb{Z}}$ are $P_{\sigma} = (x_j : j \notin \sigma)$, and their intersection is zero. Every prime contains one of them. The spectrum of $A_{\mathbb{Z}}/I_{\mathbb{Z}}$ is therefore the union of the closed subsets obtained from the coordinate components. After localization, its dimension is the maximum of the dimensions of

$$(\mathbb{Z}[x_j : j \in \sigma]/(B_{\sigma}x_{\sigma}))_{(p, x_{\sigma})}.$$

This uses a finite union of closed spectra, not an equality obtained by distributing addition of $I_{\mathbb{Z}}$ over an intersection of ideals.

On each component, Smith normal form replaces the ideal by

$$(d_1 y_1, \dots, d_n y_n), \quad 0 \neq d_1 \mid d_2 \mid \dots \mid d_n.$$

The row operations and variable changes are unimodular and preserve the ideal (p, y) . Work in the local ring

$$D = \mathbb{Z}[y_1, \dots, y_n]_{(p, y)}.$$

A prime containing all $d_i y_i$ and contracting to zero contains every y_i . If it contracts to (p) , it contains p and each y_i with $p \nmid d_i$. Conversely,

$$(y_1, \dots, y_n), \quad (p, y_i : p \nmid d_i)$$

are prime ideals containing the diagonal ideal. Their quotients have dimensions 1 and $n - r$, where $r = \text{rank}_{\mathbb{F}_p} B_\sigma$. Every containing prime contains one of these two, so the component quotient has dimension $\max\{1, n - r\}$. Taking the maximum over components proves (1).

Over $\mathbb{F}_p$, linear row and variable operations leave r independent variables in the ideal on the component, and its quotient dimension is $n - r$. Localization at the homogeneous maximal ideal preserves that dimension. The same finite-union argument proves (2). $\square$

Proof of Theorems 1.1 and 1.2. In a Noetherian local Cohen–Macaulay ring, c elements in the maximal ideal form a regular sequence exactly when the quotient dimension is the original dimension minus c [9, Tag 02JN]. At $\mathfrak{m}_p$, apply this with $c = n$ and dimension $n + 1$. Equation (1) gives regularity precisely when every vertex rank is at least $n - 1$ modulo p . Homogeneous localization detects global regularity, and the Koszul model detects surjectivity.

For a nonsingular n by n matrix with Smith invariants $d_1 \mid \cdots \mid d_n$, its finite cokernel is cyclic exactly when $|d_1| = \cdots = |d_{n-1}| = 1$. A prime dividing d_{n-1} divides at least the last two invariants, and conversely two divisible invariants give rank at most $n - 2$. The gcd of the $(n - 1)$ -minors is $|d_1 \cdots d_{n-1}|$. This proves all integral equivalences, including the empty-product convention.

Over $\mathbb{F}_p$ the Cohen–Macaulay local dimension is n . Equation (2) gives regularity exactly when all vertex matrices have full rank, or $p \nmid \det B_\sigma$. The additive Koszul model proves the prime-field assertion. No tensoring of an integral cohomology ring is substituted for this coefficient argument. $\square$

Proof of Corollary 1.3. The homotopy sequence of $G \rightarrow T \rightarrow R$ identifies $\pi_0(G)$ with $\text{coker}(B : \mathbb{Z}^m \rightarrow \mathbb{Z}^n)$. If G is connected, the character lattice sequence splits as an exact sequence of free abelian groups. Hence $H^*(BT; k) \rightarrow H^*(BG; k)$ is onto. The face-ring map $H^*(BT; k) \rightarrow A_k$ is onto as well. In the natural square with these top and left arrows and ρ_k and κ_k as right and bottom arrows, the latter two maps have the same image. The theorems apply. $\square$

For example, $B = (1, -a)$, $a > 1$, gives a connected weighted-line reduction. Integral restriction is onto, while restriction modulo any prime dividing a is not. This is a known weighted-projective phenomenon [3], not a new example claimed here.

4 Fixed sectors

Corollary 4.1. *For $g \in G$ with $Z_K^g \neq \emptyset$, put $J = \{j : g_j \neq 1\}$. The integral restriction $H_T^*(Z_K^g; \mathbb{Z}) \rightarrow H_G^*(Z_K^g; \mathbb{Z})$ is onto exactly when C_σ is cyclic for every maximal $\sigma \supseteq J$. Over $\mathbb{F}_p$, replace this by $p \nmid \det B_\sigma$ for those faces. If G is connected, the same conditions hold for $H^*(BG; k) \rightarrow H_G^*(Z_K^g; k)$. Surjectivity on all sectors is then equivalent to the corresponding condition on the identity sector.*

Proof. Nonempty fixed sets require $J \in K$. The polyhedral-product definition gives $Z_K^g = \{0\}^J \times Z_{\text{link}_K(J)}$, with the link on all remaining coordinates, including ghost vertices. Equivariantly contracting disks to zero identifies this with $Z_{\text{star}_K(J)} = (D^2)^J \times Z_{\text{link}_K(J)}$ up to T -homotopy. The star face ring is a polynomial extension of the link face ring, so is Cohen–Macaulay over the fields in question and over $\mathbb{Z}$. Its maximal faces are exactly the original maximal faces containing J , each of size n . For maximal J , the link is the empty-face complex with face ring k , so this case is included. The same dimension calculation and natural Koszul model apply. For connected G , the natural square used above again gives equality of images. The identity sector proves necessity of the all-sector criterion, and restricting the vertex list proves sufficiency. $\square$

This is a sectorwise statement about additive cohomology and the stated maps, not a calculation of the integral Chen–Ruan product.

5 A disconnected example with unit facet labels

Proposition 5.1. *There is an effective compact symplectic toric orbifold with all facet labels one and all vertex groups $C_2 \times C_2$, for which the actual integral Borel Kirwan map is onto but coordinate restriction is not.*

Proof. Let v_1, v_2, v_3 be the columns of

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = (v_1, -v_1, v_2, -v_2, v_3, -v_3).$$

The polytope $P = \{q : |\langle q, v_i \rangle| \leq 1, 1 \leq i \leq 3\}$ is a compact simple rational parallelotope. Every column is primitive, so every facet label can be one. Since $\det M = 4$, the gcd of its entries is one and the gcd of its two-minors is two, its Smith invariants are 1, 2, 2. Every vertex matrix consists of signed columns of M and has cokernel C_2^2 . The image of the full B is $M\mathbb{Z}^3$, of index four. Here $Z_K = (S^3)^3$.

In multiplicative torus coordinates (a_i, b_i) on each pair, the map factors through $c_i = a_i/b_i$ and then M . Its finite kernel in the c -torus is

$$F = \{(\epsilon_1, \epsilon_2, \epsilon_3) \in \{\pm 1\}^3 : \epsilon_1 \epsilon_2 \epsilon_3 = 1\} \cong C_2^2.$$

Indeed, the three equations for M have left sides $c_1 c_2 c_3$, $c_1 c_2^{-1} c_3$ and $c_1 c_2 c_3^{-1}$. Their ratios force $c_2^2 = c_3^2 = 1$, and then $c_1^2 = 1$. The map $(t_i; \epsilon_i) \mapsto (t_i \epsilon_i, t_i)_{i=1}^3$ therefore identifies G with $(S^1)^3 \times F$.

Put $C = H^*(BF; \mathbb{Z})$, retaining its odd torsion, and let w_i be the degree-two class of the i th character of F . Kunneth with the free torus factor gives $H^*(BG; \mathbb{Z}) = C[t_1, t_2, t_3]$. The representation on the i th complex coordinate pair has character classes $t_i + w_i$ and t_i , so its oriented rank-four Euler class is

$$e_i = t_i(t_i + w_i).$$

The e_i form a regular sequence: each is monic in its own variable, and successive quotients are free over the remaining coefficient ring with basis $1, t_i$ in that variable. Even-degree centrality makes this argument valid with odd and torsion coefficients in C .

The Borel construction of $(S^3)^3$ is the fiber product over BG of the three corresponding unit-sphere bundles. Their successive Gysin sequences, with injective Euler multiplications, give surjective pullbacks and identify the target as

$$H_G^*((S^3)^3; \mathbb{Z}) = C[t_1, t_2, t_3]/(e_1, e_2, e_3)$$

via $\kappa_{\mathbb{Z}}$. By Theorem 1.1, $\rho_{\mathbb{Z}}$ is not onto.

Finally, the Lie algebra of G is that of the diagonal torus in each pair. In complex coordinates (z_i^+, z_i^-) , its moment components are positive multiples of $|z_i^+|^2 + |z_i^-|^2$; a positive regular level is a product of spheres. Reduction by the connected torus first gives $(\mathbb{CP}^1)^3$, and F acts by multiplying the first homogeneous coordinate of factor i by ϵ_i . This action is faithful. Every nonidentity element has two negative entries, so it fixes no generic point of a toric divisor. There is no global ineffective stabilizer or generic codimension-one isotropy, confirming the unit labels. Every toric vertex is fixed by all of F . This establishes the claimed effective symplectic realization. $\square$

In particular, primitivity of the columns does not imply $B(\mathbb{Z}^m) = \mathbb{Z}^n$. Unit labels cannot replace that hypothesis in Corollary 1.3; compare the connectedness and cyclicity discussions in [3, Section 9]. We make no claim of novelty for the general phenomenon of disconnected toric kernels, or to have exhaustively settled the historical scope of those discussions.

6 Checks and scope

The accompanying standard-library Python checks enumerate 2,112 nonsingular two-by-two matrices with entries from -3 to 3 , and 236 nonsingular matrices among 240 seeded three-by-three trials. They compare determinantal divisors with ranks at determinant primes. Thirteen finite Koszul fixtures include a non-Cohen–Macaulay control with unimodular vertices but positive rational homology, showing that the algebraic hypothesis cannot simply be dropped. Normal and optimized Python runs agree. Separate checks verify all eight vertex matrices and the complete four-element kernel in Proposition 5.1. The adjugate identity with determinant four bounds that enumeration to fourth roots of unity. These computations are regression evidence, not the general proofs.

The results completely establish the stated criteria and counterexample. They do not classify the actual Kirwan map for arbitrary disconnected reducing groups, coarse-space cohomology, or integral Chen–Ruan products. A bounded primary-source search does not certify absolute priority; the antecedents above remain essential. AI assistance was used in developing arguments, checking algebra, searching the literature and preparing the manuscript. The work was self-audited in the research workflow; no independent peer review or proof-assistant certification is claimed.

References

- [1] American Institute of Mathematics, *Moment maps and surjectivity in various geometries*, workshop problem list, question 10.3 and comment 10.4. <https://aimath.org/WWN/momentmaps/momentmaps.pdf>.
- [2] M. Franz, The cohomology rings of smooth toric varieties and quotients of moment-angle complexes, *Geom. Topol.* **25** (2021), 2109–2144. doi:10.2140/gt.2021.25.2109. Author version: [arXiv:1907.04791v4](https://arxiv.org/abs/1907.04791v4).
- [3] R. Goldin, T. S. Holm and A. Knutson, Orbifold cohomology of torus quotients, *Duke Math. J.* **139** (2007), 89–139. <https://math.gmu.edu/~rgoldin/Articles/OrbifoldCohomology.pdf>.
- [4] T. Holm and T. Matsumura, Equivariant cohomology for Hamiltonian torus actions on symplectic orbifolds, *Transform. Groups* **17** (2012), 717–746. [arXiv:1008.3315v2](https://arxiv.org/abs/1008.3315v2).
- [5] T. Holm, *Integral cohomology of symplectic quotients*, CMS Winter Meeting, 2006, talk abstract based on joint work with S. Tolman. <https://www2.cms.math.ca/Reunions/hiver06/abs/pdf/pg-th.pdf>.
- [6] S. Luo, T. Matsumura and W. F. Moore, Moment angle complexes and big Cohen–Macaulayness, *Algebr. Geom. Topol.* **14** (2014), 379–406. doi:10.2140/agt.2014.14.379.
- [7] D. Pomerleano and C. Teleman, *Integral Kirwan surjectivity*, preprint, version 2, 2025. [arXiv:2410.06197v2](https://arxiv.org/abs/2410.06197v2).
- [8] R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.* **54** (1975), 135–142. doi:10.1002/sapm1975542135.
- [9] The Stacks Project Authors, *The Stacks Project*, tags 045J and 02JN, accessed 10 October 2026.