Parallel Nilpotent Endomorphisms Without Parallel Null Vectors

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Abstract

We construct a closed complete flat pseudo-Riemannian manifold of dimension 16 and signature $(8,8)$ carrying a nonzero parallel self-adjoint endomorphism N with $N^2 = 0$, although neither the manifold nor any connected double cover admits a nonzero parallel vector field. This gives a negative answer, as stated, to Question 12.0.3 in the Burns–Matveev list of open problems about geodesics. The construction starts with a compact flat Riemannian manifold whose holonomy has odd order and zero fixed space, doubles its Euclidean representation, equips the double with the split metric, and sets $N(u,v) = (0,u)$. Odd-order holonomy survives every index-two subgroup. An explicit input is the free $(\mathbb{Z}/3)^2$ quotient of an Eisenstein four-torus constructed by Bauer and Gleissner. We also show that when the top image of a parallel nilpotent has rank one, the manifold or its orientation double cover does carry a parallel null vector; in particular, the original question is affirmative in Lorentzian and co-Lorentzian signature.

1 Introduction

Burns and Matveev ask whether a closed pseudo-Riemannian manifold carrying a nonzero parallel self-adjoint nilpotent $(1,1)$-tensor must itself, or after passing to a double cover, carry a nonzero lightlike parallel vector field [4, Question 12.0.3]. Local algebras of parallel endomorphisms, including self-adjoint nilpotents, were studied systematically by Boubel [2]. In Lorentzian dimension three, Boubel and Mounoud analyzed affine transformations in the presence of a parallel lightlike vector field [3]. The unrestricted global question has a counterexample in neutral signature.

Our mechanism is a neutral doubling of an ordinary compact flat manifold. It separates the existence of a parallel tensor, which only requires an endomorphism in the commutant of holonomy, from the existence of a parallel vector, which requires a holonomy-fixed vector.

Theorem 1.1. *There exists a closed complete flat pseudo-Riemannian* 16*-manifold of signature* $(8,8)$ *carrying a nonzero parallel self-adjoint square-zero endomorphism, such that neither the manifold nor any connected double cover has a nonzero parallel vector field. Consequently it has no nonzero parallel lightlike vector field on either level.*

The example answers the literal universal question. It makes no minimality, indecomposability, or weak-irreducibility claim. Section 4 records the complementary rank-one boundary.

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2 Neutral doubling of odd holonomy

Let E be a Euclidean vector space and let $X = E/\Gamma$ be a compact flat Riemannian n-manifold. Write Λ for the translation lattice in its Bieberbach group and

$$H = \{Q_\gamma : \gamma \in \Gamma\} \leq O(E)$$

for its finite linear holonomy, where $\gamma x = Q_\gamma x + a_\gamma$.

Theorem 2.1 (Odd-holonomy doubling)**.** *If $|H|$ is odd and $E^H = \{0\}$, then there is a closed complete flat pseudo-Riemannian manifold $(\widehat{M}, G)$ of signature (n, n) with a nonzero parallel G-self-adjoint endomorphism N satisfying $N^2 = 0$, while neither $\widehat{M}$ nor any connected double cover has a nonzero parallel vector field.*

Proof. For $\gamma \in \Gamma$ and $\lambda \in \Lambda$, define an affine map of $E \oplus E$ by

$$T_{\gamma,\lambda}(x, y) = \big(Q_\gamma x + a_\gamma,\ Q_\gamma y + \lambda\big). \tag{2.1}$$

These transformations form a group $\widehat{\Gamma}$, because

$$T_{\gamma,\lambda} T_{\delta,\mu} = T_{\gamma\delta,\, \lambda + Q_\gamma \mu}. \tag{2.2}$$

Its subgroup with identity linear part is $\Lambda \oplus \Lambda$, acting by translations. Thus $\widehat{\Gamma}$ is discrete and cocompact, and its linear holonomy is

$$\widehat{H} = \{\operatorname{diag}(Q, Q) : Q \in H\}. \tag{2.3}$$

The action is free: a fixed point of $T_{\gamma,\lambda}$ would make x a fixed point of γ on E; freeness of the Bieberbach action forces γ to be the identity, after which the second coordinate forces $\lambda = 0$. Hence

$$\widehat{M} = (E \oplus E)/\widehat{\Gamma}$$

is a closed smooth manifold.

Equip $E \oplus E$ with the split bilinear form

$$G\big((u, v), (u', v')\big) = \langle u, v' \rangle + \langle v, u' \rangle. \tag{2.4}$$

Its matrix is $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, so its signature is (n, n). Every $\operatorname{diag}(Q, Q)$ preserves G, and therefore G descends to a complete flat metric on $\widehat{M}$.

Define

$$N(u, v) = (0, u). \tag{2.5}$$

It is constant and commutes with every element of $\widehat{H}$, hence descends and is parallel. It is nonzero, $N^2 = 0$, and

$$G(N(u, v), (u', v')) = \langle u, u' \rangle = G((u, v), N(u', v')),$$

so it is self-adjoint.

A parallel vector field on a complete flat quotient lifts to a constant vector fixed by the linear holonomy. Here

$$(E \oplus E)^{\widehat{H}} = E^H \oplus E^H = \{0\}. \tag{2.6}$$

Thus $\widehat{M}$ has no nonzero parallel vector.

Now let $K \le \widehat{\Gamma}$ have index two. It is normal. If $\ell : \widehat{\Gamma} \to H$ is the linear-part map, then $\ell(K)$ is normal in H, and the natural map $\widehat{\Gamma}/K \to H/\ell(K)$ is surjective. Therefore

$$[H : \ell(K)] \mid 2.$$

The same index divides the odd integer $|H|$, so it equals one. Every connected double cover consequently has the same full linear holonomy H, and (2.6) still excludes a nonzero parallel vector. $\square$

3 An explicit eight-dimensional input

Let $\zeta_3^2 + \zeta_3 + 1 = 0$,

$$E_0 = \mathbb{C}/\mathbb{Z}[\zeta_3], \qquad \tau = \frac{1 + 2\zeta_3}{3}.$$

Bauer and Gleissner construct a free affine action of $G_0 \cong (\mathbb{Z}/3)^2$ on E_0^4 [1, Example 5.6]. Its two generators can be written

$$\begin{aligned} A(z) &= \operatorname{diag}(1, \zeta_3, \zeta_3, \zeta_3^2)z + \tau(1, 2, 0, 1), \\ B(z) &= \operatorname{diag}(\zeta_3, \zeta_3, 1, \zeta_3)z + \tau(0, 0, 2, 0). \end{aligned} \tag{3.1}$$

The maps commute and have order three. For completeness, freeness is visible directly. For each nonzero A^aB^b, the coordinate in the second column below has identity linear part and a nonzero translation in E_0:

(a, b)	certificate coordinate
$(1, 0), (2, 0)$	1
$(0, 1), (0, 2)$	3
$(1, 1), (2, 2)$	4
$(1, 2), (2, 1)$	2

(3.2)

Such an affine map cannot have a fixed point, so the action is free.

The analytic representation is

$$\rho(a, b) = \operatorname{diag}\big(\zeta_3^b, \zeta_3^{a+b}, \zeta_3^a, \zeta_3^{2a+b}\big). \tag{3.3}$$

Its four characters are $(0, 1), (1, 1), (1, 0), (2, 1)$ in the variables (a, b). None is trivial; hence the fixed space in $\mathbb{C}^4$, and therefore in the underlying real eight-dimensional representation, is zero. Their kernels also have trivial intersection, so the holonomy is faithfully G_0 and has order nine.

The quotient $X = E_0^4/G_0$ is a compact flat Riemannian eight-manifold; see also [1, Remarks 5.8 and 5.10]. Applying Theorem 2.1 to its underlying real holonomy produces a dimension-16, signature-$(8, 8)$ example and proves Theorem 1.1.

4 The rank-one boundary

Proposition 4.1. *Let (M, g) carry a nonzero parallel self-adjoint nilpotent endomorphism A. If the image of the top nonzero power of A has rank one, then M or the double cover orienting that line bundle carries a nonzero parallel null vector. Consequently the Burns–Matveev question is affirmative in Lorentzian and co-Lorentzian signature.*

Proof. Let r be the nilpotency index and put $B = A^{r-1}$. Then $B \neq 0$, $B^2 = 0$, and

$$g(Bx, By) = g(x, B^2 y) = 0.$$

Thus $L = \operatorname{im} B$ is a parallel totally isotropic bundle. Since B is self-adjoint, $\ker B = L^\perp$. The formula

$$q(Bx, By) = g(x, By) \tag{4.1}$$

is well-defined and gives a parallel symmetric form on L. If $q(Bx, By) = 0$ for every y, then self-adjointness gives $g(Bx, y) = g(x, By) = 0$ for every y, hence $Bx = 0$; thus q is nondegenerate. If L has rank one, its holonomy lies in $O(q) = \{\pm 1\}$. It is already trivial when L is oriented; otherwise the connected orientation double cover kills the sign. On that level, L has a nonzero parallel section, which is null because L is totally isotropic.

In Lorentzian or co-Lorentzian signature every totally isotropic subspace has dimension at most one. Hence the hypothesis applies. □

The counterexample in Section 3 escapes Proposition 4.1 because $\operatorname{im} N$ has rank eight. No claim is made here about minimal dimension or about variants with weakly irreducible holonomy.

Null-Kähler geometry provides nearby square-zero parallel structures, but its standard compatibility condition is skew-adjoint rather than the self-adjoint condition considered here [5].

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