A Compactness Obstruction to Linear Growth Along Null Geodesics

Alper Ferudun*

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Abstract

Let (M, g) be a compact semi-Riemannian manifold of indefinite signature whose null geodesics are complete. We prove that no C^1 one-form η can have the property that, along every nonconstant affinely parametrized null geodesic $\gamma : \mathbb{R} \to M$, the function $\eta(\dot{\gamma})$ is affine with nonzero slope. This gives a negative answer to Question 9.2.1 in the Burns–Matveev list of open problems about geodesics. The proof normalizes the null cone by an auxiliary Riemannian metric. Compactness then gives a uniform lower bound for the quadratic form $(\nabla_v \eta)(v)$ on normalized null vectors, while homogeneity forces the speed of any fixed null geodesic to remain bounded. Compactness also bounds the norm of η, contradicting the asserted nonzero linear growth. Only null completeness, rather than full geodesic completeness, is used.

1 Introduction

Burns and Matveev ask whether a closed semi-Riemannian manifold of indefinite signature can carry a complete metric and a nontrivial one-form η such that every lightlike geodesic satisfies

$$\eta_{\gamma(t)}(\dot{\gamma}(t)) = C_1 t + C_2, \qquad C_1 \neq 0, \tag{1.1}$$

where the constants may depend on the geodesic [2, Question 9.2.1]. The parameter in (1.1) is affine. The question occurs in their discussion of projective transformations of semi-Riemannian metrics.

The literal existence question has a short compactness obstruction.

Theorem 1.1. *Let (M, g) be a compact semi-Riemannian manifold of indefinite signature whose null geodesics are complete, and let η be a C^1 one-form on M. It is impossible that every nonconstant affinely parametrized null geodesic $\gamma : \mathbb{R} \to M$ satisfy*

$$\eta_{\gamma(t)}(\dot{\gamma}(t)) = a_\gamma t + b_\gamma \qquad \textit{with } a_\gamma \neq 0. \tag{1.2}$$

In particular, the answer to Burns–Matveev Question 9.2.1 is negative.

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: `https://github.com/AlperTheKing`. Unrefereed preprint.

The argument needs only completeness of null geodesics, which is weaker than the completeness assumption in the question. It is independent of dimension and of the index of the metric.

The motivating projective Lichnerowicz problem has substantial positive rigidity results under additional hypotheses. Kiosak and Matveev proved the pseudo-Riemannian projective Lichnerowicz conjecture when the degree of mobility is at least three [3]. Bolsinov, Matveev, and Rosemann proved that every projective vector field on a closed connected Lorentzian manifold is affine [1]. Theorem 1.1 addresses only the exact nonzero-slope question (1.1); it does not replace those deeper global rigidity results.

2 The normalized null cone

Choose an auxiliary Riemannian metric h on M. Define the normalized null bundle

$$\mathcal{N}_h = \{(x, v) \in TM : g_x(v, v) = 0,\ |v|_h = 1\}. \tag{2.1}$$

It is a closed subset of the h-unit tangent bundle. Since M is compact, $\mathcal{N}_h$ is compact.

For a C^1 one-form η, let

$$A_\eta(x, v) = (\nabla_v \eta)_x(v). \tag{2.2}$$

This is continuous on TM and homogeneous of degree two in v. Along an affinely parametrized geodesic, put

$$F_\gamma(t) = \eta_{\gamma(t)}(\dot\gamma(t)).$$

Then

$$F'_\gamma(t) = A_\eta(\gamma(t), \dot\gamma(t)). \tag{2.3}$$

3 Proof of the theorem

Proof of Theorem 1.1. Assume the forbidden property. Every $(x, v) \in \mathcal{N}_h$ is the initial condition of a complete affinely parametrized null geodesic. Differentiating (1.2) at zero and using (2.3) gives

$$A_\eta(x, v) = a_{\gamma_v} \neq 0.$$

Thus the continuous function $|A_\eta|$ is positive on the compact set $\mathcal{N}_h$, and hence

$$m := \min_{\mathcal{N}_h} |A_\eta| > 0. \tag{3.1}$$

Fix one complete affine null geodesic γ and write its nonzero slope as a. Equation (2.3), now at every time, gives

$$A_\eta(\gamma(t), \dot\gamma(t)) = a. \tag{3.2}$$

Put $r(t) = |\dot\gamma(t)|_h$. The tangent of a nonconstant geodesic never vanishes, so $r(t) > 0$, and

$$w(t) = \frac{\dot\gamma(t)}{r(t)}$$

belongs to $\mathcal{N}_h$. Homogeneity, (3.1), and (3.2) imply

$$|a| = r(t)^2 |A_\eta(\gamma(t), w(t))| \geq m r(t)^2.$$

Consequently

$$|\dot\gamma(t)|_h \le \sqrt{|a|/m} \qquad (t \in \mathbb{R}). \tag{3.3}$$

Let $L = \max_M |\eta|_h$, which is finite by compactness. From (3.3),

$$|\eta_{\gamma(t)}(\dot\gamma(t))| \le L\sqrt{|a|/m} \qquad (t \in \mathbb{R}).$$

The left-hand side is therefore bounded on $\mathbb{R}$, contradicting $\eta(\dot\gamma(t)) = at + b$ with $a \neq 0$. This proves the theorem. □

4 Parameter invariance and scope

If $\tilde\gamma(t) = \gamma(ct + d)$ with $c \neq 0$, then

$$\eta(\dot{\tilde\gamma}(t)) = c\,\eta(\dot\gamma(ct + d)).$$

Thus an affine formula of slope a transforms to one of slope $c^2 a$. In particular, the condition that the slope be nonzero is independent of the affine scale chosen for a geometric null geodesic.

For a projective vector field, the differential equation recalled by Burns and Matveev makes $\eta(\dot\gamma)$ affine along each null geodesic. Theorem 1.1 forces at least one null direction with zero slope. It does not force zero slope in every null direction and therefore does not, by itself, prove the full projective Lichnerowicz–Obata conjecture in arbitrary indefinite signature.

Acknowledgments

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References

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