

Minimal Geodesics under Finite Covers

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Abstract

A minimal closed geodesic need not have a simple projection under a finite Riemannian cover. We give explicit flat double covers and primitive minimal geodesics whose primitive projections have exactly m double points, for every positive integer m . Both manifolds can be chosen orientable in every dimension at least three. These examples disprove the simple-projection clause of Theorem 2.5 in Contreras and Mazzucchelli's *Closed geodesics and the first Betti number*. The obstruction is that a deck isometry need not preserve the minimizing cohomology class. We also prove a sufficient repair: if that class is pulled back from the base, the projected geodesic is minimal and its primitive traversal is simple. The authors' non-cover perturbation theorem then applies on the base. No general entropy-density question is resolved here.

1 Minimality and the finite cover question

For a closed one-form σ on a closed Riemannian manifold (M, g) , put

$$\mathcal{A}_\sigma^g(\zeta) = \frac{1}{T} \int_0^T \left(\frac{1}{2} |\dot{\zeta}(t)|_g^2 - \sigma(\dot{\zeta}(t)) \right) dt, \quad \zeta(0) = \zeta(T). \quad (1)$$

Here $T > 0$ and ζ is a $W^{1,2}$ loop. Its action depends only on the class $c = [\sigma] \in H^1(M; \mathbb{R})$. A nonconstant closed geodesic is *c-minimal* if one period attains the infimum of (1) over all such loops and all positive periods. This is the convention of [1, section 1.2]. A closed geodesic is *simple* if its restriction to a half-open primitive period is injective.

Theorem 2.5 of [1] concerns a finite cover $p : M \rightarrow B$ and a metric h on B . Given a minimal closed geodesic γ of p^*h , its conclusion preserves γ under a small base-metric perturbation and requires its projection to be simple. The same clause occurs in arXiv:2407.02995v2, page 13. We exhibit an obstruction that is independent of the size of the perturbation: the fixed projected curve is not simple for any metric.

Theorem 1.1. *For every $n \geq 2$ there is a regular two-sheeted Riemannian cover $p : \mathbb{T}^n \rightarrow B_n$ of closed connected flat manifolds such that, for every integer $m \geq 1$, a minimal closed geodesic γ_m upstairs has the following properties:*

1. γ_m is embedded and has primitive period one.
2. $p \circ \gamma_m$ also has primitive period one and has exactly m double points, with linearly independent tangent lines.
3. If $n \geq 3$, both $\mathbb{T}^n$ and B_n are orientable.

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In particular, the simple-projection conclusion of the stated Theorem 2.5 is false.

The word “transverse” applies to these double points in dimension two. In higher dimensions we assert independent tangent lines, not transversality of two one-dimensional submanifolds.

2 The explicit flat examples

Give $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ its Euclidean metric. For $n = 2$ set

$$\iota(x, y) = (x + \tfrac{1}{2}, -y),$$

and for $n \geq 3$ set

$$\iota(x, y, z, x_4, \dots, x_n) = (x + \tfrac{1}{2}, -y, -z, x_4, \dots, x_n). \quad (2)$$

Coordinates are taken modulo one. This is an isometric involution without fixed points, since its first coordinate changes by $1/2$. Consequently $B_n = \mathbb{T}^n / \langle \iota \rangle$ is a smooth closed flat manifold and its quotient map is a regular double cover. For $n \geq 3$ the differential of ι has determinant 1, so the quotient is orientable. For $n = 2$ it is a Klein bottle.

For $m \geq 1$ define

$$v_m = (1, m, 0, \dots, 0), \quad \gamma_m(t) = (t, mt, 0, \dots, 0) \pmod{\mathbb{Z}^n}, \quad \sigma_m = dx + m dy. \quad (3)$$

The first coordinate shows that γ_m is embedded on $[0, 1)$ and has primitive period one. For every tangent vector w ,

$$\tfrac{1}{2}|w|^2 - \sigma_m(w) = \tfrac{1}{2}|w - v_m|^2 - \tfrac{1 + m^2}{2}. \quad (4)$$

Thus every loop of every positive period has action at least $-(1 + m^2)/2$, and γ_m attains this bound. In particular it is minimal for the nonzero class $[\sigma_m]$, in exactly the required global, time-averaged sense.

Write $\beta_m = p \circ \gamma_m$. Distinct parameter values t, s modulo one represent the same point of β_m precisely when $\gamma_m(t) = \iota(\gamma_m(s))$, or equivalently

$$t - s = \tfrac{1}{2} \pmod{1}, \quad m(t + s) = 0 \pmod{1}. \quad (5)$$

The collision times form the set

$$\left\{ t_j = \tfrac{j}{2m} - \tfrac{1}{4} \pmod{1} : 0 \leq j < 2m \right\}.$$

They are paired by $t_j \leftrightarrow t_{j+m} = t_j + 1/2 \pmod{1}$, where indices are taken modulo $2m$. Each pair gives one double point. There are no higher-multiplicity points, since each fiber of p has two points and γ_m is embedded. The two tangent vectors at a common lift are v_m and $d\iota(v_m) = (1, -m, 0, \dots, 0)$. Their first two coordinates have determinant $-2m$, so their lines are independent.

It remains to rule out a shorter primitive period downstairs. If $\beta_m(t + T) = \beta_m(t)$ for every t , uniqueness of a lift gives

$$\gamma_m(t + T) = d(\gamma_m(t)) \quad \text{for all } t,$$

with a single deck transformation $d \in \{1, \iota\}$. Equivalently, the choice is locally constant and $\mathbb{R}$ is connected. The case $d = \iota$ is impossible after differentiating, since $m > 0$. The identity case forces $T \in \mathbb{Z}$ from the first coordinate. Hence the primitive period remains one.

For $m = 1$, the parameters $1/4$ and $3/4$ already give a concrete double point. Simplicity of this fixed curve is independent of the metric. No change of metric can remove its independent tangent lines or make its image embedded. This completes the proof of Theorem 1.1. $\square$

3 The cohomology class cannot be ignored

An isometry d satisfies

$$\mathcal{A}_{(d^{-1})^*\sigma}^g(d \circ \zeta) = \mathcal{A}_\sigma^g(\zeta). \quad (6)$$

It therefore transports a c -minimal orbit to a $(d^{-1})^*c$ -minimal orbit. It need not preserve the minimizing class. For our involution,

$$\iota^*\sigma_m = dx - m dy, \quad \mathcal{A}_{\sigma_m}(\iota \circ \gamma_m) = \frac{-1 + 3m^2}{2} = -\frac{1 + m^2}{2} + 2m^2.$$

The translated orbit is not $[\sigma_m]$ -minimal. Nor is the average of the two periodic orbit measures a minimizing measure for that class. This invalidates the common-class premise in the deck-transformation argument of [1, proof of Theorem 2.5]. It does not contradict Mather's graph theorem.

4 A sufficient descent condition

The following proposition isolates a hypothesis under which projection does preserve minimality. It also works for nonregular finite covers.

Proposition 4.1. *Let $p : (M, p^*h) \rightarrow (B, h)$ be a finite Riemannian cover. If γ is c -minimal and $c = p^*b$ for $0 \neq b \in H^1(B; \mathbb{R})$, then $p \circ \gamma$ is b -minimal and its primitive traversal is simple. For a regular finite cover it suffices that c be invariant under the whole deck group.*

Proof. Choose a closed representative η of b and use $p^*\eta$ for c ; adding an exact form changes no closed-loop action. Let $a_B(b)$ and $a_M(p^*b)$ denote the infima in (1) downstairs and upstairs. Projection preserves the integrand and period of every closed loop, so $a_B(b) \leq a_M(p^*b)$.

Conversely, any closed loop ζ on B induces a permutation of a finite fiber. Some positive iterate ζ^k has a closed lift to M . Iteration preserves time-averaged action, as does lifting. It follows that $a_M(p^*b) \leq \mathcal{A}_\eta^h(\zeta)$ for every ζ . Consequently

$$a_B(b) = a_M(p^*b). \quad (7)$$

The projection of γ attains this common value. Removing repetitions does not change its average.

Let β denote this primitive geodesic, with period T . If $\beta(0) = \beta(s)$ for $0 < s < T$, split it into loops on $[0, s]$ and $[s, T]$. Each loop has average at least $a_B(b)$, and their period-weighted average is $a_B(b)$. Both averages must therefore equal $a_B(b)$. In particular, the first segment minimizes the action among all closed loops of fixed period s .

For any $w \in T_{\beta(0)}B$, take a variation through closed loops with endpoint variation field $V(0) = V(s) = w$. The interior first variation of the kinetic integral vanishes by the geodesic equation; its boundary term is

$$\langle \dot{\beta}(s) - \dot{\beta}(0), w \rangle_h.$$

The one-form integral has zero variation, since $d\eta = 0$ and its endpoint terms cancel. Minimality and arbitrary w give $\dot{\beta}(s) = \dot{\beta}(0)$. Uniqueness for the geodesic initial-value problem makes s a period, a contradiction. Thus β is simple.

Finally, if the cover is regular with finite deck group D and c is invariant, average a closed representative σ :

$$\bar{\sigma} = \frac{1}{|D|} \sum_{d \in D} d^*\sigma.$$

This represents c and is deck-invariant, so descends to a closed form on B . Since $c \neq 0$, its descending class is nonzero. $\square$

Corollary 4.2. *Under the assumptions of Proposition 4.1, with $\dim B \geq 2$, a metric arbitrarily C^∞ -close to h preserves the primitive projected geodesic and makes it hyperbolic with a transverse homoclinic. The original closed lift is hyperbolic for the pullback metric.*

Proof. Apply the credited non-cover Theorem A of [1] to the minimal geodesic on B . The return map along the closed lift is conjugate to a positive integral power of the base return map, which preserves hyperbolicity. $\square$

The descent condition is sufficient, not asserted necessary for every individual orbit with a simple projection. Corollary 4.2 does not supply the same generality as the original finite-cover statement. Its homoclinic construction is an application of the cited theorem, not a new dynamical argument.

5 Scope and verification

This correction arose in studying the positive-entropy density question in the AIM geodesic problem list [2, Question 8.4.1]. It does not resolve that question. Zero entropy of the displayed flat metrics would not by itself refute density, and the failure for a prescribed projected orbit does not preclude a different orbit or a different perturbation argument. Neither Theorem A nor Corollary B of [1] is refuted here; the truth of its general Corollary 2.6 is not decided by this note.

The geometry of flat quotients, the action identity, finite iteration of loops and averaging of forms are standard ingredients. A bounded primary-literature search did not identify an earlier published version of this specific correction; that does not establish absolute priority.

The accompanying standard-library Python checker uses exact rational arithmetic. It passed 30,744 assertions for 125 members of the family, with $n \in \{2, 3, 4, 5, 8\}$ and $1 \leq m \leq 25$, including the collision pairing, action gap and iteration identities. Replays in Python 3.13 and 3.12 produced identical outputs. These finite tests support the formulas; the proofs for all dimensions, all m and all finite covers are the arguments above. A second runtime is not independent review.

AI-assisted tools supported research, exact calculation, proof development and manuscript preparation. The author is responsible for the final claims and text. This preprint is self-audited and unrefereed; no independent human review or proof-assistant verification is claimed.

References

- [1] G. Contreras and M. Mazzucchelli, *Closed geodesics and the first Betti number*, Proc. London Math. Soc. **131** (2025), no. 3, e70085. <https://doi.org/10.1112/plms.70085>. Preprint: <https://arxiv.org/abs/2407.02995v2>.
- [2] K. Burns and V. S. Matveev, *Open Problems and Questions about Geodesics*, AIM problem list, section 8.4, Question 8.4.1. <https://aimath.org/pastworkshops/geodesicsproblems.pdf>.