

Expansive Symmetry Breaking at a Five-Vertex Linkage Singularity

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Abstract

We give an explicit planar framework with five distinct vertices spanning the plane and six bars whose locally one-dimensional configuration set has two analytic branches at a reflection-symmetric configuration. Both outgoing branches are expansive and immediately break the reflection. Every nonedge distance strictly increases, as follows from formulas valid on an entire interval. The initial bars overlap, but every strictly positive-time configuration in the stated interval is noncrossing. The starting infinitesimal flex space modulo rigid motions has dimension two, not one. Five is the smallest number of vertices for immediate expansive symmetry breaking among connected simple frameworks with an injective full-span initial placement and a one-dimensional configuration germ. In contrast, a finite relabeling symmetry fixes a regular one-dimensional squared-distance curve pointwise whenever a nonconstant expansive path leaves a fixed point. We also record analytic continuation and a fixed-frame formulation. The example applies classical conic four-bar kinematics and distinguishes singular local dimension from a regular one-degree-of-freedom mechanism; it is not a counterexample under the latter hypothesis or a noncrossing initial-configuration requirement.

1 Symmetry and the meaning of one degree of freedom

Let $G = (V, E)$ be a finite simple graph. A framework is a labeled placement $p = (p_i)_{i \in V}$ in $\mathbb{R}^d$, with each edge representing a bar. A motion preserves the edge lengths. It is *expansive* if all labeled pairwise distances are nondecreasing. We allow intersections between bars unless noncrossing is explicitly required.

An initial symmetry is a permutation $\sigma \in \text{Aut}(G)$ realized by a Euclidean isometry at p . We say it is preserved at a later configuration q if some Euclidean isometry realizes the *same permutation* at q . This definition is invariant under a change of rigid reference frame. A motion immediately breaks a symmetry if that permutation is not realized at arbitrarily small positive times.

Schulze's Problem 14 in the AIM linkage workshop list asks whether a symmetric framework with a one-degree-of-freedom expansive mechanism preserves its symmetry [1]. The short formulation does not specify whether one degree of freedom means a regular curve, infinitesimal dimension one, or a possibly singular configuration set of local dimension one. These distinctions matter. The construction below answers the permissive singular interpretation negatively, while the regular distance-curve statement is positive. The overlapping starting bars exclude an interpretation requiring an initially noncrossing linkage. No unconditional closure of the workshop question is asserted.

Connelly, Demaine and Rote construct a particular symmetry-preserving expansive motion for arc-and-cycle sets [3, Section 5.1]. Their selection argument uses uniqueness of the chosen

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infinitesimal motion; it is not an assertion about every singular linkage branch. Schulze’s symmetric infinitesimal-to-finite flex theorem assumes regularity in the symmetry-restricted configuration space [4, Theorems 4.2–4.3]. The example below fails that regularity condition. The workshop report’s negative answer concerning a pseudo-triangulation algorithm refers to the neighboring Problem 13, not to the present formulation [2].

The four-bar core of our example has cyclic lengths 4, 3, 2, 3 and lies in the classical conic I family. He and Guest give general parametrizations and describe the common limit of its branches [5, Sections 2, 11 and 20]. We do not claim a new classification of four-bar singularities. Our purpose is the explicit all-pairs expansive application, its full-span realization, and the sharp five-vertex lower bound under the stated assumptions. A tangent orientation obstruction underlying Section 2 is also recorded in the frozen machine-generated research report associated with the dataset entry AIM-GEOMETRY-0201 [6]; we give a distance-curve formulation that does not require a nonzero initial time derivative or full affine span.

2 Regular distance curves

For a placement p , write

$$F(p) = (|p_i - p_j|^2)_{\{i,j\} \subset V}.$$

Equal F vectors characterize Euclidean congruence, even if the affine span is smaller than $\mathbb{R}^d$. A finite vertex-permutation group Γ acts on these vectors by coordinate permutations. A permutation is realized by a congruence at p precisely when it fixes $F(p)$. We use the set of F vectors of all realizations with the prescribed bar lengths as the *distance configuration set*.

Theorem 2.1. *Let x be fixed by Γ . Suppose the germ of the distance configuration set at x is a Γ -invariant C^1 embedded one-manifold, possibly with boundary. If a continuous, nonconstant expansive path leaves x , then Γ fixes a neighborhood of x pointwise. If this germ belongs to a connected Γ -invariant regular one-dimensional component C , then Γ fixes all of C .*

Proof. Choose distinct path points $y_k \rightarrow x$ on one side of a regular coordinate chart. The unit secants $(y_k - x)/|y_k - x|$ converge to a unit tangent vector $a \geq 0$, coordinatewise. Each group element acts orthogonally on the ambient distance coordinates, preserves the tangent line, and therefore sends a to a or $-a$. The latter is impossible: a coordinate permutation sends a nonzero nonnegative vector to another such vector.

Choose a small invariant connected neighborhood of x in the curve, by taking the component containing x of the intersection of the finitely many images of a coordinate neighborhood. Each group element restricts to an increasing finite-order interval homeomorphism. Such a map is the identity: if $h(y) > y$, all successive iterates increase and cannot return to y ; similarly if $h(y) < y$. This proves the local conclusion. On a connected regular component the orientation sign is constant. An orientation-preserving finite-order homeomorphism of an interval is the identity; for a circle, remove the fixed point and apply the interval argument. At a boundary point the inward tangent half-line is preserved, and the same argument applies. $\square$

The original time parametrization may have zero velocity, or be flat to every finite order, at its initial point. The hypothesis concerns the regular embedded distance curve, not that parametrization.

Corollary 2.2. *Let $p(t)$ be a real-analytic framework motion on a connected interval. If a fixed vertex-permutation group is realized on a nonempty open subinterval, it is realized throughout the motion, including at later singular configurations.*

Proof. For every $g \in \Gamma$ and pair i, j , the real-analytic function $|p_i(t) - p_j(t)|^2 - |p_{gi}(t) - p_{gj}(t)|^2$ vanishes on an open interval and hence everywhere. $\square$

This does not allow arbitrary switching between analytic branches, nor does it cover a motion starting at the singularity below.

3 An explicit expansive symmetry-breaking singularity

Use five vertices with initial positions

$$A = (-2, 0), \quad B = (2, 0), \quad C = (-1, 0), \quad D = (1, 0), \quad E = (0, 1), \quad (1)$$

and the six bars

$$AB, AD, AE, BC, BE, CD. \quad (2)$$

Their respective lengths are $4, 3, \sqrt{5}, 3, \sqrt{5}, 2$. Reflection $S(x, y) = (-x, y)$ induces $\sigma = (A B)(C D)$, fixing E .

Theorem 3.1. *The framework (1)–(2) has an expansive motion for $0 \leq a < \pi/6$ that strictly increases every nonedge distance and breaks σ for every $a > 0$. Its configuration germ modulo congruences consists of two smooth analytic branches meeting with distinct tangents. Thus its real local dimension is one, but its initial nontrivial infinitesimal flex space has dimension two. The initial bars overlap; every indicated positive-time configuration is a noncrossing straight-line framework.*

Proof. Keep the triangle ABE fixed. For $0 < a < \pi/6$, set

$$c = \cos a, \quad b = \arccos \frac{1 + 3c^2}{4c}, \quad u = a + b, \quad v = a - b, \quad (3)$$

and define

$$C(a) = (2 - 3 \cos u, -3 \sin u), \quad D(a) = (-2 + 3 \cos v, -3 \sin v). \quad (4)$$

At zero take $b = 0$. The identities

$$\cos b - c = \frac{1 - c^2}{4c} > 0, \quad 1 - \cos b = \frac{(1 - c)(3c - 1)}{4c} > 0$$

give $0 < b < a$ and $0 < v < u < 2a < \pi/3$. Differentiation yields

$$b' = \frac{3c^2 - 1}{c\sqrt{9c^2 - 1}}, \quad 0 < b' < 1, \quad (5)$$

where the upper bound follows from

$$c^2(9c^2 - 1) - (3c^2 - 1)^2 = 5c^2 - 1 > 0.$$

Thus $u', v' > 0$.

The triangle is fixed, and (4) gives $|BC| = |AD| = 3$. The remaining bar condition is

$$\begin{aligned} |C - D|^2 &= 34 - 24(\cos u + \cos v) + 18 \cos(u + v) = 4 \\ \iff 2 + 6 \cos^2 a - 8 \cos a \cos b &= 0, \end{aligned} \quad (6)$$

which follows from (3). These are exact identities on the entire interval.

The only nonedges are AC, BD, CE, DE , with squared lengths

$$|A - C|^2 = 25 - 24 \cos u, \quad |B - D|^2 = 25 - 24 \cos v, \quad (7)$$

$$|C - E|^2 = 14 - 12 \cos u + 6 \sin u, \quad |D - E|^2 = 14 - 12 \cos v + 6 \sin v. \quad (8)$$

The angle derivatives are $24 \sin \theta$ and $12 \sin \theta + 6 \cos \theta$, both positive for $0 < \theta < \pi/3$. Together with $u', v' > 0$ this proves strict increase for positive a and nondecrease from zero. In particular, all vertices remain distinct.

Any isometry implementing σ must exchange A, B and fix E . Since these three points span the plane, it must be S . But $u > v$ and (8) give $|C - E| > |D - E|$, which excludes σ in any rigid reference frame. The graph has no other nonidentity automorphism: A, B are its only degree-three vertices, E is their unique common neighbor, and C, D lie on the remaining three-edge path between them.

We next check the local configuration set. With ABE fixed, the two circles for C, D have local coordinates u, v . Their remaining constraint is

$$f(u, v) = 5 - 4(\cos u + \cos v) + 3 \cos(u + v) = 0. \quad (9)$$

Its quadratic term is $(u^2 + v^2)/2 - 3uv$, which is nondegenerate and indefinite. More explicitly, in $a = (u + v)/2$, $b = (u - v)/2$ coordinates the solutions near zero are exactly

$$b = \pm \psi(a), \quad \psi(a) = \arcsin \frac{\sin a \sqrt{9 \cos^2 a - 1}}{4 \cos a}. \quad (10)$$

The function ψ is real analytic near zero and has $\psi'(0) = 1/\sqrt{2}$. The two tangent lines are distinct, and reflection interchanges the branches. At a nonzero nearby point the b derivative in (6) is $8 \cos a \sin b \neq 0$, so the configuration curve is regular. The initial full rigidity matrix has rank five: the triangle gives three independent constraints, the two circle constraints give two more, and the remaining first-order equation in u, v vanishes. The infinitesimal dimension modulo rigid motions is therefore $10 - 3 - 5 = 2$. The two positive- a half-branches are both expansive, one being the reflection of the other.

Finally, D is strictly inside the clockwise triangle ABC for $a > 0$. The three oriented area numerators are

$$\begin{aligned} \text{orient}(A, B, D) &= -12 \sin v, \\ \text{orient}(B, C, D) &= 9 \sin(2a) - 12 \sin u, \\ \text{orient}(C, A, D) &= 12 \sin v - 9 \sin(2a). \end{aligned}$$

They are negative because

$$\sin v < \frac{3}{4} \sin(2a) < \sin u.$$

Indeed, with $r = \sqrt{9c^2 - 1}$, the sine addition formulas reduce these inequalities to $-r < (3c^2 - 1)/c < r$, already contained in (5). Thus AD and CD lie inside ABC apart from their endpoints A, C and do not cross its boundary. The edges AE, BE lie above AB , whereas C, D lie below it. There are no nonincident bar intersections. At zero, by contrast, AB contains C, D and several cycle bars overlap. $\square$

Remark 3.2. A finite symmetric branch would have $u = v$, in which case (9) becomes $2(3 \cos u - 1)(\cos u - 1) = 0$. Near zero it has only the constant solution. The fully symmetric infinitesimal direction therefore does not integrate to a nonconstant symmetric finite motion. Averaging the two finite reflected motions also fails: it removes the nonzero vertical difference between C, D and shortens their bar below length two.

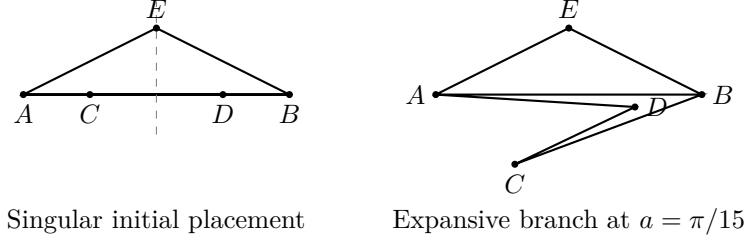


Figure 1: The six bars are those in (2). Several overlap on the initial horizontal line. At positive time D is inside ABC ; the framework is noncrossing but no longer reflection-symmetric.

4 Five vertices are minimal for immediate breaking

Lemma 4.1. *At a full-span placement in $\mathbb{R}^d$, a velocity preserving all squared distances to first order is an infinitesimal rigid motion. Consequently, on a smooth Euclidean slice the map F is an immersion.*

Proof. Translate one point and its velocity to zero and choose d position vectors forming a basis. Define A by mapping those vectors to their velocities. The all-pairs equations give $p_i \cdot Ap_j + p_j \cdot Ap_i = 0$, so A is skew-symmetric. For any other vertex k , the equations with the basis vectors imply $p_i \cdot (u_k - Ap_k) = 0$ for every basis vector. Hence $u_k = Ap_k$. Restoring the translation proves the assertion. $\square$

Theorem 4.2. *Among connected simple planar frameworks with distinct initial vertices spanning $\mathbb{R}^2$ and a configuration germ of real local dimension one modulo Euclidean congruences, five vertices are necessary and sufficient for immediate expansive loss of an initial vertex-permutation symmetry.*

Proof. Theorem 3.1 proves sufficiency. Full span requires at least three vertices. A connected three-vertex graph is a path or a triangle. The triangle is rigid. For the path, fix one bar and let the third vertex move on its circle. At a noncollinear configuration its distance image is a regular curve, so Theorem 2.1 excludes immediate expansive symmetry loss.

Consider four vertices. A connected three-edge graph is a tree. After fixing one bar, its two remaining bar angles are independent local coordinates, so its configuration germ has dimension two. A connected four-edge graph is either a four-cycle or a triangle with a pendant edge.

For a four-cycle, a nonzero self-stress has no zero edge coefficient: equilibrium at a degree-two vertex would otherwise force its other coefficient to vanish, and this propagates around the cycle. Nonzero stress consequently makes the two incident edge vectors collinear at every vertex, forcing all four vertices to be collinear. Full span excludes this. The four rigidity rows are independent, and the implicit function theorem on a Euclidean slice gives a smooth one-dimensional curve. Lemma 4.1 makes its distance image a regular embedded curve locally. Apply Theorem 2.1.

For a triangle with a pendant edge, fix a positive-length triangle edge. Its third vertex lies in a finite intersection of two circles with distinct centers and is locally constant, even when the triangle is collinear. The pendant vertex moves on a positive-radius circle about its neighbor. By full span and Lemma 4.1, the distance map has nonzero derivative on that circle. Its image is a regular embedded curve, and the same theorem applies. This argument concerns the real configuration set; it does not assume that a collinear triangle has a maximal-rank rigidity matrix.

A five-edge graph on four vertices is K_4 minus an edge. Its two triangles share a positive-length edge; after fixing that edge, each remaining vertex lies in a finite two-circle intersection. There

is no one-dimensional configuration germ. Adding the sixth edge cannot create one. These cases exhaust the graphs. $\square$

The lower bound concerns loss at arbitrarily small positive times. It makes no claim about a symmetry first lost only after a later singularity, or about disconnected, coincident-vertex, or non-full-span starting configurations.

5 A fixed symmetry group in the laboratory frame

The distance formulation permits a moving rigid frame. The following standard finite-group argument shows that, under full span, this does not weaken the fixed-frame formulation of a symmetry-preserving C^1 motion.

Proposition 5.1. *Suppose a C^1 motion $p(t)$ has full affine span throughout an interval and realizes the same finite permutation group Γ at every time. There is a C^1 rigid change of frame, equal to the identity at a chosen time t_0 , after which the initial Euclidean symmetries realize that group at every time.*

Proof. Let z be the centroid, let Y have columns $p_i - z$, and let Z_g have columns $p_{gi} - z$. The uniquely determined orthogonal representation is

$$A_g = Z_g Y^T (Y Y^T)^{-1}.$$

Full span ensures invertibility and C^1 dependence. Set $L_g = A'_g A_g^{-1}$ and $X = |\Gamma|^{-1} \sum_g L_g$; all are skew-symmetric. Differentiation of the representation law gives $L_{hg} = L_h + A_h L_g A_h^{-1}$. Average over g to obtain

$$L_h = X - A_h X A_h^{-1}, \quad A'_h = [X, A_h].$$

Solve $Q' = XQ$, $Q(t_0) = I$. Then $Q \in \text{SO}(d)$ and $Q^{-1} A_g Q = A_g(t_0)$. The reframed points

$$p_i^*(t) = Q(t)^{-1} (p_i(t) - z(t)) + z(t_0)$$

are acted on by the fixed initial isometries $y \mapsto A_g(t_0)(y - z(t_0)) + z(t_0)$. Rigid changes of frame preserve every distance and hence expansiveness. $\square$

The full-span condition is imposed along the entire interval in this proposition. Corollary 2.2 concerns only the distance symmetry and does not require this fixed-frame construction through rank drops.

6 Scope and reproducibility

The complete statements proved here are the regular distance-curve theorem, the singular expansive example, its noncrossing opening, and the five-vertex minimality assertion within its explicit class. The example is not a regular one-dimensional infinitesimal mechanism at its starting point and is not initially noncrossing. These are not technicalities that can be omitted when relating the result to the AIM question.

The accompanying standard-library Python programs use exact rational and quadratic-field arithmetic. They check bar lengths and all-pairs derivatives at 251 rational half-angle parameter values, test the strict interior orientations for 250 positive values, and enumerate the 43 connected labeled graphs on two through four vertices. They also include collinear-rank and failed-averaging

controls. The finite checks supplement the interval and arbitrary-configuration proofs, rather than replace them.

AI tools assisted with research, derivation, computation and drafting. The manuscript is self-audited and unrefereed; no independent peer review, proof-assistant certification or absolute priority is claimed.

References

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