Angle-Data Reconstruction for Triangulated Polyhedral Surfaces and a Genus Deficit

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Abstract

An AIM problem asks for a characterization of the face-angle and dihedral- angle data of a triangulated polyhedral surface in $\mathbb{R}^3$ and conjectures that the realizable data have dimension $E-1$ in every genus. We use the realization convention later made explicit by Hempel: a labeled, simplexwise-linear, simplexwise-injective map of a consistently oriented triangulated closed surface, modulo similarities. The combined face-angle and oriented-dihedral map is injective and has a local real-analytic left inverse at every nondegenerate realization. Its actual image therefore has dimension

$$3V-7=E-1-6g.$$

Thus the AIM formula is correct for the sphere and fails in every positive genus. The embedded Császár torus gives the concrete count 14, rather than 20. We also give a finite necessary-and-sufficient realizability test: intrinsic sine-law compatibility followed by vertex and non-tree-hinge closure in a dual-spanning-tree development. At generic realizations, Fogelsanger rigidity shows that face angles alone already have rank $E-1-6g$, identifying the missing $6g$ as global extrinsic closure codimension. This note is unrefereed and makes no absolute priority claim.

1 Introduction

Problem 8 in the 2014 AIM list *Configuration spaces of linkages* asks which collections of face and dihedral angles are realized by simplicial polyhedra with fixed combinatorics, and conjectures that the resulting space has dimension $E-1$ [1]. The workshop report makes the genus quantifier explicit: the formula was recorded for $g=0$ and conjectured for general genus [2].

Hempel subsequently developed intrinsic triangle constraints and local polyhedral-cone constraints for this problem [5, 6]. Her sufficiency result is simply connected. Moreover, the current arXiv v3 states that its smooth-dimension calculation is conditional on the Elimination Pattern Conjecture after an error was found in the earlier lemma [6]. We do not use that elimination system. Instead, we work directly with the actual vertex-coordinate realization space and reconstruct coordinates from angle data along a dual tree.

Our main conclusion is that the proposed general-genus formula confuses two spaces. Intrinsic edge-length ratios, equivalently compatible face angles, have dimension $E-1$. Actual spatial face-plus-dihedral data have dimension $E-1-6g$. The extra conditions are global closure conditions around the handles.

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2 The realization and angle spaces

Let $K = (K_0, K_1, K_2)$ be a finite connected, consistently oriented abstract simplicial complex whose realization is a closed orientable surface of genus g. Write

$$V = |K_0|, \qquad E = |K_1|, \qquad F = |K_2|.$$

Every edge belongs to two faces, so $3F = 2E$, and Euler's formula gives

$$E = 3V - 6 + 6g. \tag{1}$$

Following Hempel [5, Definition 3], let U_K be the open subset of $(\mathbb{R}^3)^V$ consisting of labeled vertex maps whose affine extension is injective on each simplex. Equivalently, every triangular face is noncollinear. Global injectivity is not required.

Let

$$\mathrm{Sim}^+(3) = \{x \mapsto \lambda R x + t : \lambda > 0,\ R \in SO(3),\ t \in \mathbb{R}^3\}.$$

This seven-dimensional group acts freely on U_K: a stabilizer fixes the three noncollinear labeled vertices of any face and is therefore the identity. Put

$$\mathcal{X}_K = U_K / \mathrm{Sim}^+(3), \qquad \dim \mathcal{X}_K = 3V - 7. \tag{2}$$

For each corner (f, v), let $\sigma_{f,v} \in (0, \pi)$ be its face angle. For each edge e, retain the oriented dihedral as a point of S^1, represented by $(\cos \delta_e, \sin \delta_e)$. The combined angle map is

$$\mathcal{A}_K : \mathcal{X}_K \longrightarrow (0, \pi)^{3F} \times (S^1)^E, \qquad [p] \longmapsto (\sigma(p), \delta(p)). \tag{3}$$

It is real analytic. The circle encoding avoids branch trouble at coplanar hinges.

3 Analytic reconstruction

Theorem 3.1. *The map $\mathcal{A}_K$ is injective and has a real-analytic local left inverse at every point. In particular, it is an immersion and every local branch of its actual image has dimension*

$$\dim \mathcal{A}_K(\mathcal{X}_K) = 3V - 7 = E - 1 - 6g. \tag{4}$$

The same local formula holds on the embedded realization locus.

Proof. Choose one consistently oriented root face (i, j, k). Every similarity orbit has a unique representative satisfying

$$p_i = 0, \qquad p_j = (1, 0, 0), \qquad p_k = (x, y, 0), \quad y > 0. \tag{5}$$

This is a smooth global slice for the quotient.

Choose a spanning tree T in the dual graph of faces, rooted at (i, j, k). Fix one actual angle datum. In a neighborhood of it, reconstruct the root triangle analytically from its face angles and the normalization $|p_i - p_j| = 1$. Traverse T. Suppose a child face is reached across a placed edge. Its face angles and the known common-edge length determine its other two edge lengths by the sine law. The oriented dihedral determines the unique placement of its oriented copy by rotation around the common edge.

If the third abstract vertex of the child has not appeared before, assign it this newly computed position. If it has appeared, retain its previously assigned position and use the resulting ordered face frame in subsequent steps. The second rule can ignore some coordinates of an inconsistent ambient input; that is harmless. At the fixed actual datum, and throughout a sufficiently small neighborhood, every finitely many face frames used by the algorithm stays nondegenerate. Sine-law ratios, normalized vectors, cross products, and rotations represented by $(\cos\delta, \sin\delta)$ are analytic. The procedure therefore defines an analytic map R_T on an ambient neighborhood of the datum.

If the input is the angle datum of an actual normalized realization, every new vertex is placed at its original coordinate. Induction over T gives

$$R_T \circ \mathcal{A}_K = \mathrm{id}_{\mathcal{X}_K}\,. \tag{6}$$

Consequently $DR_T D\mathcal{A}_K = I$, so $D\mathcal{A}_K$ is injective. The analytic constant-rank theorem and (2) give the first equality in (4). Combining with (1) gives the second.

The same induction proves global injectivity. Given two realizations with the same angle data, align their root faces by an orientation-preserving similarity. Each newly encountered vertex is then forced to coincide in the two realizations, so all labeled vertices agree.

For a finite simplicial complex, global embeddings form an open subset near every embedded nondegenerate realization. Restricting the preceding local argument to that open subset does not change the dimension. □

Remark 3.2. Set-theoretic injectivity would not by itself imply injective differential. The local left inverse (6) is the essential rank statement. Flexible polyhedra cause no contradiction: a nontrivial flex preserves face shapes but must change at least one oriented dihedral.

4 A finite realizability criterion

We now characterize arbitrary proposed data. For an edge e in a face f, write $\mathrm{opp}_f(e)$ for the opposite corner.

Proposition 4.1. *A collection $\sigma_{f,v} \in (0,\pi)$ is the face-angle collection of an intrinsic Euclidean polyhedral metric on K, up to common scale, if and only if:*

1. *$\sum_{v\in f} \sigma_{f,v} = \pi$ for every face f;*
2. *there are positive numbers r_f, one for each face, such that for every edge $e = f \cap f'$,*

$$r_f \sin\sigma_{f,\mathrm{opp}_f(e)} = r_{f'} \sin\sigma_{f',\mathrm{opp}_{f'}(e)}. \tag{7}$$

The space of such angle data is an analytic manifold of dimension $E - 1$.

Proof. For a Euclidean triangle the side opposite α has length $r\sin\alpha$, where r is twice the circumradius. Thus (7) is exactly shared-edge length compatibility. It is necessary, and when it holds it defines consistent positive edge lengths and hence glued Euclidean triangles. The dual graph is connected, so the r_f are unique up to common multiplication. Equivalently, the sine ratios have product one around every dual cycle.

Positive edge lengths satisfying all strict face triangle inequalities form an open subset of $\mathbb{R}^E_{>0}$. Quotienting by common scale gives dimension $E - 1$. The cosine law maps this quotient analytically and injectively to the face angles; the sine-law propagation above is an analytic inverse on the image. □

Assume the conditions in proposition 4.1, normalize one edge length, choose a dual spanning tree T, and place a root face. Develop a separate oriented copy of each child face across every tree edge, using the prescribed dihedral. Denote the developed occurrence of abstract vertex v in face f by $x_{f,v} \in \mathbb{R}^3$.

Theorem 4.2. *The combined data* (σ, δ) *are realized by a polyhedron in Hempel's sense if and only if proposition 4.1 holds and the following finite closure conditions hold:*

1. *for every vertex* v*, all occurrences* $x_{f,v}$ *are equal;*
2. *for every dual edge outside* T*, the two developed incident faces have the prescribed oriented dihedral.*

Proof. Developing an actual realization proves necessity. Conversely, vertex closure defines one point p_v for every abstract vertex. Each developed face is then the triangle spanned by its three p_v and has the prescribed face angles. Dihedrals on tree edges hold by construction; those on the remaining edges hold by the second closure condition. The resulting affine map on each face is nondegenerate and realizes all of the data. □

This criterion is exact but deliberately redundant. It can be written using finite analytic equations in the sine and cosine coordinates. Global embedding adds the finite nonintersection conditions for pairs of simplices.

5 The generic $6g$ closure codimension

Theorem 3.1 already gives the dimension of the combined data at every realization and uses no rigidity theorem. Generic rigidity explains the same deficit in the projection to face angles alone.

Let $\mathcal{B}_K : \mathcal{X}_K \to (0, \pi)^{3F}$ be the face-angle map. Suppose an infinitesimal motion $u = (u_v)$ fixes every face angle. In one face, the sine law shows that the logarithmic derivatives of its three edge lengths are equal; call the common value c_f. Adjacent faces share an edge, and dual connectivity therefore makes every c_f equal to one global number c. After subtracting the infinitesimal scaling motion $p_v \mapsto cp_v$, all edge lengths are infinitesimally fixed.

Fogelsanger proved that the graph of every connected triangulated closed surface is generically rigid in $\mathbb{R}^3$; a modern account is given in [3]. Hence, at a generic realization, the remaining motion is Euclidean. After quotienting by similarities,

$$\operatorname{rank} D\mathcal{B}_K = 3V - 7 = E - 1 - 6g. \tag{8}$$

By proposition 4.1, the ambient intrinsic face-angle space has dimension $E - 1$. Thus the actual generic face-angle locus has codimension $6g$ inside it. These are global spatial closure conditions, not additional intrinsic triangle-gluing conditions.

6 The Császár torus

Császár constructed an embedded triangulated torus with seven vertices, twenty-one edges, and fourteen triangular faces [4]. Applying theorem 3.1 gives

$$\dim \mathcal{A}_K(\mathcal{X}_K) = 3 \cdot 7 - 7 = 14, \qquad E - 1 = 20.$$

This is an embedded counterexample to the conjectured positive-genus dimension, with the predicted deficit six.

For reproducibility, the accompanying exact checker builds the cyclic seven-vertex combinatoric from the two face orbits

$$\{i, i+1, i+3\}, \qquad \{i, i+2, i+3\} \quad (i \bmod 7).$$

It verifies all edge incidences, vertex links, orientability, and genus. At seven rational moment-curve points it computes the Jacobian of all normalized squared edge lengths over $\mathbb{Q}$ and obtains rank 14. Since compatible face angles analytically determine normalized edge lengths, this is also an independent finite rank certificate for the face-angle map at that simplexwise-nondegenerate realization.

7 Scope and prior-work calibration

The AIM item itself says that clarification is needed. Our theorem uses the definition later written by the proposer and therefore allows self-intersections. It also applies locally to embedded realizations. It does not classify connected components of the embedded locus, knotting, or convexity; closed convex surfaces are necessarily genus zero.

The tree-closure equations are finite and necessary and sufficient, but not minimal. The $6g$ statement is a rank count and should not be read as a claim that one canonical subset of exactly $6g$ scalar equations works at all points.

Targeted searches through August 29, 2026 checked the AIM sources, Hempel's thesis and current v3, intrinsic polyhedral metric work including [7], rigidity sources, the Császár paper, and broader realization- space literature. We did not locate the positive-genus formula $E - 1 - 6g$ or the present dual-tree closure result. The reconstruction is elementary enough that it may be folklore, so no absolute priority claim is made.

Verification

An exact-arithmetic checker accompanies the source. It verifies the Császár combinatorics and a rank-14 rational Jacobian certificate. Computational and language-model assistance was used for source discovery, proof auditing, and manuscript preparation. The named author retains responsibility for every claim. This manuscript has not been peer reviewed.

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