Complex Sectional Curvature Blow-Up under Circle Cheeger Collapse

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September 2, 2026

Abstract

An AIM problem asks whether complex sectional curvatures remain uniformly bounded below in a collapsing circle Cheeger deformation. At a fixed point whose normal circle representation contains rotation blocks of speeds $a, b > 0$, we derive an exact fixed-point formula. A totally isotropic complex two-plane has

$$K_{\mathbb{C}}(g_\varepsilon) = K_{\mathbb{C}}(g) - \frac{ab}{\varepsilon^2}.$$

Thus every fixed component of real codimension at least four forces complex sectional curvature to diverge to $-\infty$. For the standard weight-$(1,1)$ action on the unit round S^4, the value at either fixed pole is $1 - \varepsilon^{-2}$. With only one rotation block, the singular curvature operator is instead rank one and positive semidefinite. The AIM workshop report records the corresponding negative curvature-operator phenomenon but does not identify a complex decomposable direction; the observation here is that its two real negative directions combine into a decomposable totally isotropic bivector. The note is unrefereed and makes no priority claim for this elementary calculation.

1 Introduction

Let a circle act isometrically on a Riemannian manifold (M, g). Give S^1 the metric for which its period-2π generator has length ε, and let g_ε be the quotient metric in

$$(M, g) \times (\varepsilon S^1) \longrightarrow M \times_{S^1} (\varepsilon S^1) \cong M. \tag{1}$$

This is the circle Cheeger deformation in the parameterization used in the AIM list *Geometric flows and Riemannian geometry*. Problem 4.3 asks whether the curvature operators of g_ε are bounded below independently of ε, and Problem 4.4 asks: "For the above problem, what about complex sectional curvature?" [1]. The construction also appears naturally in Lott's study of collapse with a lower curvature-operator bound [3].

The AIM workshop report records a sharp-looking orbit-type distinction for the curvature operator. It reports a uniform lower bound for almost-free actions and for fixed sets of codimension two. For a fixed set of larger codimension, calculations in flat $\mathbb{R}^n$ produce an eigenvalue tending to $-\infty$; the report suggests passing from the normal representation to the general case [2]. A negative curvature-operator eigenvector need not be decomposable, even after complexification, so that statement alone does not answer Problem 4.4.

We show that the negative four-dimensional block does contain a complex decomposable direction. In fact it contains a totally isotropic one, so the counterexample works both for complex sectional

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Unrefereed preprint.

curvature in the sense of Ni–Wolfson [4] and for the narrower condition that tests only totally isotropic planes. The resulting compact example starts from the round S^4 and retains nonnegative real sectional curvature throughout the Cheeger deformation, while its complex sectional curvature is unbounded below as $\varepsilon \downarrow 0$.

2 The fixed-point curvature tensor

We use the curvature-tensor convention for which $R(X, Y, X, Y) = \sec(X \wedge Y)$ when X, Y are orthonormal. Extend g and R complex bilinearly. For a complex plane with a unitary basis Z, W, its complex sectional curvature is

$$K_{\mathbb{C}}(Z, W) = R(Z, W, \overline{Z}, \overline{W}). \tag{2}$$

For arbitrary linearly independent Z, W, the right-hand side is divided by

$$g(Z, \overline{Z})g(W, \overline{W}) - g(Z, \overline{W})g(W, \overline{Z}).$$

This agrees with ordinary sectional curvature on real planes.

Let K denote the action field of the period-2π generator. If p is a fixed point, put

$$A = (\nabla K)_p, \qquad \omega(X, Y) = g(AX, Y).$$

The Killing equation makes A skew-adjoint and ω a two-form.

Lemma 2.1 (Fixed-point formula)**.** *At a fixed point p, the quotient metric agrees with the original metric, $g_\varepsilon|_p = g|_p$, and*

$$R^\varepsilon(X, Y, Z, W) = R(X, Y, Z, W) + \varepsilon^{-2}T_\omega(X, Y, Z, W), \tag{3}$$

where

$$\begin{aligned} T_\omega(X, Y, Z, W) = 2\omega(X, Y)\omega(Z, W) &+ \omega(X, Z)\omega(Y, W) \\ &- \omega(X, W)\omega(Y, Z). \end{aligned} \tag{4}$$

In particular, $T_\omega(X, Y, X, Y) = 3\omega(X, Y)^2$.

Proof. Under the standard identification of the quotient in (1) with M, horizontal lifting gives the exact metric formula

$$g_\varepsilon(X, Y) = g(X, Y) - \frac{g(X, K)g(Y, K)}{\varepsilon^2 + |K|^2}. \tag{5}$$

Choose g-normal coordinates at p. Since $K(p) = 0$ and $K(x) = Ax + O(|x|^2)$, the metric difference has vanishing first derivative and

$$\partial_k\partial_l(g_\varepsilon - g)_{ij}\big|_p = -\varepsilon^{-2}(A_{ik}A_{jl} + A_{il}A_{jk}). \tag{6}$$

Substitution into the normal-coordinate curvature identity

$$R_{ijkl} = \tfrac{1}{2}(g_{jk,il} - g_{jl,ik} - g_{ik,jl} + g_{il,jk})$$

gives (3)–(4). Equivalently, (4) is the polarized form of O'Neill's horizontal sectional curvature equation [5]. □

3 A totally isotropic negative direction

At a fixed point, the normal circle representation is an orthogonal sum of real two-dimensional rotation blocks. Orient two nonzero blocks and choose orthonormal vectors so that

$$Ax_1 = ay_1, \quad Ay_1 = -ax_1, \qquad Ax_2 = by_2, \quad Ay_2 = -bx_2, \tag{7}$$

with $a, b > 0$.

Theorem 3.1. *In the situation of* (7), *the unitary vectors*

$$Z = \frac{x_1 + iy_1}{\sqrt{2}}, \qquad W = \frac{x_2 + iy_2}{\sqrt{2}}$$

span a totally isotropic complex plane and satisfy

$$\boxed{K_{\mathbb{C},g_\varepsilon}(Z, W) = K_{\mathbb{C},g}(Z, W) - \frac{ab}{\varepsilon^2}.} \tag{8}$$

Consequently, if the action has a fixed component of real codimension at least four, then

$$\inf_{p,\sigma} K_{\mathbb{C},g_\varepsilon}(\sigma) \longrightarrow -\infty \qquad (\varepsilon \downarrow 0).$$

Proof. The bilinear complexification of g gives

$$g(Z, Z) = g(W, W) = g(Z, W) = 0,$$

so the plane is totally isotropic. From (7),

$$\omega(Z, W) = \omega(Z, \overline{W}) = 0, \qquad \omega(Z, \overline{Z}) = -ia, \quad \omega(W, \overline{W}) = -ib.$$

Therefore (4) gives

$$T_\omega(Z, W, \overline{Z}, \overline{W}) = (-ia)(-ib) = -ab.$$

Equation (8) follows from lemma 2.1. A fixed component of codimension at least four has at least two nonzero normal rotation blocks, while the original curvature term in (8) is independent of ε. □

The link with the workshop's real curvature-operator calculation is particularly transparent. On the four-dimensional sum of the two blocks, put

$$u = x_1 \wedge x_2 - y_1 \wedge y_2, \qquad v = x_1 \wedge y_2 + y_1 \wedge x_2.$$

Both are real eigenvectors of the singular curvature operator with eigenvalue $-ab$, but neither is decomposable. Their complex combination is

$$u + iv = (x_1 + iy_1) \wedge (x_2 + iy_2), \tag{9}$$

which is precisely the decomposable bivector used in theorem 3.1.

Proposition 3.2 (One-block boundary)**.** *If* rank $A \leq 2$ *at a fixed point, the singular curvature operator defined by T_ω is positive semidefinite. If A has one nonzero block of speed a, it has rank one and its only nonzero eigenvalue is $3a^2$.*

Proof. Choose an orthonormal basis with $\omega = a\, x^* \wedge y^*$. Formula (4) vanishes on all curvature-operator entries except the $x \wedge y$ diagonal entry, which equals $3a^2$. □

4 A compact counterexample and scope

Let

$$S^4 = \{(r, z_1, z_2) \in \mathbb{R} \oplus \mathbb{C}^2 : r^2 + |z_1|^2 + |z_2|^2 = 1\}$$

carry its round metric of sectional curvature one, and act by

$$e^{it}(r, z_1, z_2) = (r, e^{it} z_1, e^{it} z_2).$$

The poles $(\pm 1, 0, 0)$ are fixed, and their normal representations have two blocks with $a = b = 1$. A constant-curvature-one tensor has complex sectional curvature one on every unitary complex plane. Hence theorem 3.1 immediately gives the promised closed example.

Corollary 4.1. *At either fixed pole of the weight-$(1,1)$ action on the unit round S^4, there is a totally isotropic complex plane with*

$$\boxed{K_{\mathbb{C},g_\varepsilon} = 1 - \varepsilon^{-2}.} \tag{10}$$

Thus the complex sectional curvatures of this Cheeger family have no lower bound independent of ε.

The real sectional curvatures remain nonnegative, since (1) is a Riemannian quotient of the product of the round sphere and a flat circle. In the flat linear model on $\mathbb{C}^2$ with weights a, b, the same computation gives the exact value $-|ab|\varepsilon^{-2}$ at the origin, while every real sectional curvature there is nonnegative. This isolates the mixed complex direction rather than a negative real plane.

The conclusion is a negative answer to the uniform-lower-bound reading of AIM Problem 4.4. It does not reprove the global positive statements for almost-free actions or for codimension-two fixed sets that are recorded in the workshop report. The targeted literature audit located the real curvature-operator warning in that report, the general quotient context in Lott, and the standard complex-curvature definition in Ni–Wolfson, but no source explicitly stating (8), (9), or (10). Since the deduction from the fixed-point tensor is short and may be folklore, no assertion of priority is intended.

Reproducibility. An exact Gaussian-rational checker verifies the algebraic curvature symmetries, the coordinate Hessian formula, the two-dimensional negative eigenspace, total isotropy, and (10). A separate implementation starts from (5), differentiates it using exact second-order rational jets, and independently obtains $-ab/\varepsilon^2$.

The calculation, targeted literature triage, manuscript drafting, and checker construction were carried out with assistance from OpenAI Codex. The submitting human author must review the source record, proof, citations, and final manuscript and assumes responsibility for every claim.

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