

Smooth Convergence of Inverse Gauss Curvature Flow on the Two-Sphere

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Abstract

We prove smooth convergence of the normalized inverse Gauss curvature flow on the two-sphere for every smooth initial conformal metric of positive Gauss curvature. No smallness or symmetry assumption is imposed, and convergence takes place in the original conformal coordinates. The argument upgrades the known global existence and weak convergence: the curvature-weighted distance estimate of Gursky and Streets, together with Darvas’s endpoint comparison, controls background energy; curvature entropy then controls the conformal factor through the spherical Green kernel. Two maximum-principle estimates give time-uniform upper and lower curvature bounds. An exact change of variables to inverse Monge–Ampère flow supplies higher regularity, and distance contraction selects a single round limit. The result concerns the explicit two-dimensional question, not its higher-dimensional extensions.

1 Statement and previously established inputs

Let g_0 be the round metric of Gauss curvature one on $\mathbb{S}^2$, and put $V = \text{Area}(g_0) = 4\pi$. All Laplacians below have negative spectrum. For $g = e^{2u}g_0$, write

$$f = 1 - \Delta_0 u = K e^{2u}, \quad A = \int_{\mathbb{S}^2} e^{2u} dA_0, \quad b = \frac{V}{A}. \quad (1)$$

Thus b is the average Gauss curvature. We study

$$\partial_t u = \frac{b}{K} - 1. \quad (2)$$

Theorem 1.1. *For every $u_0 \in C^\infty(\mathbb{S}^2)$ with $1 - \Delta_0 u_0 > 0$, the solution of (2) exists for all $t \geq 0$ and converges in $C^\infty(\mathbb{S}^2)$ to a smooth u_∞ satisfying $K_\infty = b_\infty > 0$. In particular, $e^{2u_\infty}g_0$ is round up to scale and a conformal automorphism. No time-dependent change of coordinates is required.*

The question appears as Problem 3.3 in the AIM workshop list [1]; it is recorded as AIM-GEOMETRY-0171 in the frozen UnsolvedMath v1.6.0 collection. Theorem 1.1 answers its spherical component in the smooth topology. The additional question about higher dimensions is not addressed here.

Gursky and Streets [2, Propositions 4.6, 4.7, 4.25 and 4.26] established global existence, contraction of a curvature-weighted path distance, monotonicity of curvature entropy and weak H^1

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convergence. Their published 2019 proof of the H^1 bound invokes Darvas [3, Theorem 3]; the 2015 arXiv version has a different concluding argument. We use the published estimates and supply a uniform smoothness upgrade. Watkin's stability theorem [5, Theorem 5.2.5] has smallness and decay hypotheses. The regularity estimates of Collins–Hisamoto–Takahashi [4, Proposition 2.7 and Corollary 2.8] will be applied after proving an exact transformation of the flow. No absolute priority claim is made.

All constants below may depend on u_0 and g_0 , but not on an upper time endpoint. Global existence and positivity are supplied by [2].

2 Background energy and entropy

For $f > 0$, consider the path metric whose tangent norm is

$$\|v\|_u^2 = \int_{\mathbb{S}^2} v^2 f dA_0,$$

and denote its distance by d . Length contraction in [2] implies distance contraction: evolve any smooth almost minimizing initial path, use smooth dependence and global existence on each finite time interval, and take the infimum of its initial length. A smooth minimizing geodesic is not needed. Since $u = 0$ is stationary,

$$d(u(t), 0) \leq d(u_0, 0). \quad (3)$$

Lemma 2.1. *Along the flow, the quantities*

$$\int u^2 dA_0, \quad \int u^2 f dA_0, \quad \int |\nabla_0 u|^2 dA_0, \quad |\bar{u}|, \quad \bar{u} = V^{-1} \int u dA_0,$$

are uniformly bounded.

Proof. Identify $\mathbb{S}^2$ with $\mathbb{CP}^1$, let $\omega_0 = dA_0$, and use $i\partial\bar{\partial}v = (\Delta_0 v)\omega_0/2$. The potential $\phi = -2u$ has

$$\omega_\phi = \omega_0 + i\partial\bar{\partial}\phi = f\omega_0 > 0.$$

The normalized Mabuchi L^2 norm of $-2v$ is $(4/V) \int v^2 f dA_0$. Darvas's endpoint comparison [3, Theorem 3; arXiv Theorem 5.5], together with (3), therefore gives

$$\int u^2 dA_0 + \int u^2 f dA_0 \leq C.$$

This comparison is on the space of potentials, not a quotient by constants. Since $\int f dA_0 = V$, integration by parts gives

$$\int |\nabla_0 u|^2 dA_0 = \int u(f - 1) dA_0 \leq \sqrt{V} \left[\left(\int u^2 f dA_0 \right)^{1/2} + \left(\int u^2 dA_0 \right)^{1/2} \right].$$

Cauchy–Schwarz also bounds $|\bar{u}|$. □

Define the conserved quantity and curvature entropy

$$J = \int_{\mathbb{S}^2} (|\nabla_0 u|^2 + 2u) dA_0, \quad E = \int_{\mathbb{S}^2} K \log K dA_g.$$

Direct calculation yields

$$J' = 2 \int u_t f dA_0 = 2 \int (b - K) dA_g = 0, \quad (4)$$

$$K_t = -b\Delta_g(K^{-1}) + 2(K - b), \quad (5)$$

$$\partial_t dA_g = 2(b/K - 1)dA_g. \quad (6)$$

In particular, integration by parts gives the entropy identity

$$E' = -b \int K |\nabla_g(K^{-1})|^2 dA_g \leq 0. \quad (7)$$

Indeed, the differentiated integrand has Laplacian term $-b(1 + \log K)\Delta_g(K^{-1})$ and reaction term $2(K - b)$; the latter integrates to zero. This also recovers [2, Proposition 4.7].

For the background probability measure $d\mu = dA_0/V$, set

$$H = \int f \log f d\mu = \frac{1}{V} \left(E + 2 \int u dA_0 + 2 \int |\nabla_0 u|^2 dA_0 \right). \quad (8)$$

Since f has mean one, $H \geq 0$. Lemma 2.1 and (7) show that $H \leq C$ uniformly.

3 Pointwise control from the Green kernel

Lemma 3.1. *If $f = 1 - \Delta_0 u > 0$ and $H = \int f \log f d\mu$, then*

$$\bar{u} - 1 \leq u(x) \leq \bar{u} + 2H + 2 \log 2 - 1 \quad (x \in \mathbb{S}^2). \quad (9)$$

Proof. The mean-zero Green function of $-\Delta_0$ is

$$G(x, y) = -\frac{1}{V} \log \frac{1 - x \cdot y}{2} - \frac{1}{V}.$$

It satisfies $-\Delta_y G = \delta_x - 1/V$ and $G \geq -1/V$. For $0 < a < V$, rotational invariance and the substitution $s = (1 - x \cdot y)/2$ give

$$Z_a := \int e^{aG(x, y)} d\mu(y) = \frac{e^{-a/V}}{1 - a/V}. \quad (10)$$

Green representation gives $u(x) - \bar{u} = V \int G(x, y) f(y) d\mu(y)$, so the lower bound follows. Nonnegativity of relative entropy with respect to e^{aG}/Z_a gives

$$a \int G f d\mu \leq H + \log Z_a.$$

One may first replace G by $\min(G, N)$ and then pass to the limit; the fixed smooth density and the logarithmic pole are integrable. Taking $a = V/2$ proves the upper bound. $\square$

It follows that there are constants m, M with

$$m \leq u \leq M, \quad 0 < b_0 := e^{-2M} \leq b \leq b_1 := e^{-2m}. \quad (11)$$

Thus no L^1 -to- L^∞ elliptic estimate is assumed.

4 Two curvature barriers

Write $r = e^{-2u}$ and introduce

$$L = \partial_t - \frac{b}{K^2} \Delta_g.$$

The identity $\Delta_g u = r - K$ and (5) imply

$$Lu = -1 + \frac{2b}{K} - \frac{br}{K^2}, \quad (12)$$

$$L \log K = 2 - \frac{2b}{K} - \frac{b}{K^2} |\nabla_g \log K|^2, \quad (13)$$

$$L(K^{-1}) = -2K^{-1} + 2bK^{-2}. \quad (14)$$

Proposition 4.1. *There are positive constants k_0, k_1 such that $k_0 \leq K(x, t) \leq k_1$ for every x and $t \geq 0$.*

Proof. For $W = \log K + 3u$, equations (12)–(13) give

$$LW = -1 + \frac{4b}{K} - \frac{3br}{K^2} - \frac{b}{K^2} |\nabla_g \log K|^2 \leq -1 + \frac{4b_1}{K}.$$

If W exceeded both its initial maximum and $\log(8b_1) + 3M$, then at a positive-time space-time maximum it would have $K > 8b_1$ and $LW < -1/2$, a contradiction. A first-crossing argument with a strictly increased threshold removes equality cases. Hence

$$K \leq \exp \left(\max_{\mathbb{S}^2} \{ \sup W(0), \log(8b_1) + 3M \} - 3m \right).$$

For the other bound, put $q = K^{-1}$, $B = 4e^{2M}$ and $Z = q + Bu$. Then

$$LZ = b(2 - Br)q^2 + (2Bb - 2)q - B \leq -2b_0q^2 + 2Bb_1q - B.$$

For $q \geq Q := \max\{1, 2Bb_1/b_0\}$ the right side is at most $-b_0q^2 - B < 0$. The same maximum principle gives

$$q \leq \max_{\mathbb{S}^2} \{ \sup Z(0), Q + BM \} - Bm.$$

Both arguments apply on arbitrary finite time intervals of the known smooth positive-curvature solution. Their constants do not depend on the interval endpoint; uniform parabolicity at infinite time was not assumed. $\square$

5 Higher regularity and one limiting metric

Set $\tau = 2t$ and $\phi(x, \tau) = -2u(x, \tau/2)$. With the complex conventions from Lemma 2.1, $\omega_\phi = f\omega_0$. Since $\text{Ric}(\omega_0) = \omega_0$, we have

$$\text{Ric}(\omega_\phi) - \omega_\phi = i\partial\bar{\partial}(-\log f + 2u).$$

Thus the normalized Ricci potential is $\rho = \log b - \log K$:

$$\int e^\rho \omega_\phi = b \int e^{2u} dA_0 = V, \quad \partial_\tau \phi = 1 - e^\rho.$$

This is precisely the inverse Monge–Ampère flow. Equations (11) and Proposition 4.1 give bounded $|\phi|$ and uniform equivalence $c\omega_0 \leq \omega_\phi = Ke^{2u}\omega_0 \leq C\omega_0$. Consequently [4, Proposition 2.7 and Corollary 2.8] give uniform spatial C^k bounds for every k . These interior estimates can be used on fixed-length sliding intervals, with the fixed smooth initial solution controlling an initial compact interval. The regularity statements do not assume absence of holomorphic vector fields. In particular, they apply to $\mathbb{CP}^1$; no such restriction is dropped from a convergence theorem.

It remains to show that the whole trajectory has one limit. Put

$$F = J - V \log(A/V), \quad D = \int \frac{(K-b)^2}{K} dA_g.$$

Using (4) and $A' = 2b \int K^{-1} dA_g - 2A$ gives

$$F' = -2D. \tag{15}$$

The uniform bound on u bounds F below; hence $\int_0^\infty D dt < \infty$. Choose $t_j \rightarrow \infty$ with $D(t_j) \rightarrow 0$. Uniform higher regularity gives a subsequence with $u(t_j) \rightarrow u_\infty$ in C^∞ . The lower curvature bound permits passage to the limit in D , proving $K_\infty = b_\infty > 0$. Thus u_∞ is stationary.

The straight segment between two positive-curvature potentials v, w stays in the positive cone, since $1 - \Delta_0 u$ is affine. Its tangent is constant and its curvature measure has total mass V . Therefore

$$d(v, w) \leq \sqrt{V} \|v - w\|_\infty. \tag{16}$$

Contraction toward u_∞ makes $d(u(t), u_\infty)$ nonincreasing. Equation (16) and the convergent subsequence force this distance to tend to zero. Every smooth cluster point v consequently has $d(v, u_\infty) = 0$, by the triangle inequality and (16). Darvas’s estimate makes the distance nondegenerate, so $v = u_\infty$. Precompactness in each C^k now proves convergence of the whole trajectory, completing the proof of Theorem 1.1.

6 Scope and boundary examples

The stationary Möbius factors

$$u_a(z) = \tfrac{1}{2} \log(1 - a^2) - \log(1 - az), \quad 0 < a < 1,$$

where z is the vertical coordinate, satisfy $K = 1$ and $E = 0$. Their extrema become unbounded as $a \rightarrow 1$. Thus curvature entropy alone cannot bound the conformal factor, and the limiting potential need not be constant. Background energy control is essential in (8). The logarithmic Green estimate is specific to real dimension two.

The Fano convergence discussion in [4, Remark 4.8] distinguishes weak from smooth convergence. The convergence criteria in [6, Theorem 1.1] assume absence of holomorphic vector fields and therefore do not directly apply to $\mathbb{CP}^1$. Our argument uses their predecessor’s regularity estimates only after establishing the needed pointwise bounds. No prescribed nonconstant target curvature or higher-dimensional flow is covered.

The accompanying exact-arithmetic script checks evolution identities, barrier signs, the Möbius family and normalization factors on finite rational test sets. These checks supplement the analytic proof and do not constitute a proof-assistant verification. AI assistance was used for mathematical exploration, literature comparison, drafting and computational checks; the author is responsible for the claims. This preprint has not undergone independent peer review.

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