

Exact Small-Softness Stabilization of Euclidean Lattice Packings

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Abstract

For a Euclidean lattice whose nonzero vectors have length at least two, consider the density of the union of balls of radius $1 + \lambda$ centered at its points. We prove that, in every fixed dimension $d \geq 2$ and for all sufficiently small $\lambda > 0$, the maximizing lattices are exactly the densest hard-sphere lattices with the smallest kissing number among the hard-optimal lattices. The proof compares a linear local covolume margin with a vanishing Lipschitz constant for pair overlaps, and uses Mahler compactness to exclude all other competitors. In dimension three this proves the lattice alternative of Bezdek–Lángi’s small-softness FCC conjecture and gives the optimal density $\pi(1 + 3\lambda - 6\lambda^2 - 5\lambda^3)/(3\sqrt{2})$. The global softness threshold is existential. The result does not treat general nonlattice packings. This manuscript is unrefereed.

1 Statement and relation to the source problem

Let B^d be the closed Euclidean unit ball and $v_d = \text{vol}(B^d)$. A full-rank lattice $L \subset \mathbb{R}^d$ is *admissible* if $|v| \geq 2$ for every $v \in L \setminus \{0\}$. Its unit balls then form a hard packing. Write $V(L)$ for its covolume and define its soft density by

$$\rho_\lambda(L) = \frac{\text{vol}((L + (1 + \lambda)B^d) \cap F)}{V(L)}, \quad \lambda \geq 0, \quad (1)$$

where F is a fundamental parallelepiped. Periodicity makes this independent of F and equal to the large-ball density. Overlaps are counted once. In particular, $\rho_0(L) = v_d/V(L)$.

The AIM soft-packing list asks whether soft-packing results stabilize to hard-packing results as softness tends to zero [1]. An explicit extremal instance is Conjecture 2 of Bezdek and Lángi [3]: in dimension three, is the expanded FCC packing optimal for all sufficiently small positive softness, among lattice packings and, separately, among all packings? Their Theorem 3 proves local FCC optimality at every fixed softness. We prove the full lattice alternative of the conjecture. The nonlattice alternative and the broader, open-ended AIM question are not resolved here.

Put

$$V_d = \inf\{V(L) : L \text{ is admissible}\}, \quad D_d^{\text{lat}}(\lambda) = \sup\{\rho_\lambda(L) : L \text{ is admissible}\}.$$

Let $\mathcal{H}_d$ denote the orthogonal equivalence classes of lattices with covolume V_d . Below we prove directly that this set is nonempty and finite. For $L \in \mathcal{H}_d$, let $\tau(L)$ be the number of vectors of length two, counting both signs, and set $\tau_d = \min_{L \in \mathcal{H}_d} \tau(L)$.

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For two radius- r balls whose centers are distance s apart, write

$$\ell_d(r, s) = \begin{cases} 2v_{d-1} \int_{s/2}^r (r^2 - t^2)^{(d-1)/2} dt, & 0 \leq s \leq 2r, \\ 0, & s \geq 2r. \end{cases} \quad (2)$$

Theorem 1.1. *For every fixed $d \geq 2$, there is $\varepsilon_d > 0$ such that, for $0 < \lambda < \varepsilon_d$, the maximizers of ρ_λ over all admissible lattices are exactly the orthogonal images of the members of $\mathcal{H}_d$ with kissing number τ_d . In particular,*

$$D_d^{\text{lat}}(\lambda) = \frac{v_d(1 + \lambda)^d - \frac{1}{2}\tau_d\ell_d(1 + \lambda, 2)}{V_d}. \quad (3)$$

The assertion includes attainment of the soft optimum; no classification of the hard-optimal lattices is assumed.

Corollary 1.2. *For all sufficiently small $\lambda > 0$, the FCC lattice of minimum vector length two is the unique maximizer in dimension three, up to orthogonal transformations, and*

$$D_3^{\text{lat}}(\lambda) = \frac{\pi}{3\sqrt{2}}(1 + 3\lambda - 6\lambda^2 - 5\lambda^3). \quad (4)$$

The threshold in Theorem 1.1 is not numerically specified. Section 5 gives an explicit *local* FCC certificate; its numbers are not asserted to be a global threshold. The local rigidity used below is an elementary Gram-form expression of classical hard-lattice extremality. The no-triple-intersection threshold and its inclusion-exclusion use already appear in [2]. The new conclusion claimed here is exact global small-softness selection, with a uniform local comparison and a compactness argument that applies to every lattice competitor.

2 Pair overlaps and their vanishing slope

Lemma 2.1. *If $1 \leq r < 2/\sqrt{3}$ and L is admissible, then*

$$\rho_{r-1}(L) = \frac{N_r(L)}{V(L)}, \quad N_r(L) = v_d r^d - \frac{1}{2} \sum_{v \in L \setminus \{0\}} \ell_d(r, |v|). \quad (5)$$

Moreover, $N_r(L) \geq v_d$. For $d \geq 2$ and $s \geq 2$,

$$0 \leq \ell_d(r, 2) - \ell_d(r, s) \leq v_{d-1}(r^2 - 1)^{(d-1)/2}(s - 2). \quad (6)$$

Proof. Suppose three radius- r balls with mutually separated centers x_1, x_2, x_3 share a point z , and put $m = (x_1 + x_2 + x_3)/3$. Then

$$4 \leq \frac{1}{3} \sum_{i < j} |x_i - x_j|^2 = \sum_i |x_i - m|^2 \leq \sum_i |x_i - z|^2 \leq 3r^2.$$

Thus no triple intersections occur below $2/\sqrt{3}$. Integrate pairwise inclusion-exclusion over F . For the single-ball terms, translating F through L gives the integral over the whole space, namely $v_d r^d$. For any relative displacement $v \neq 0$, the same translation argument gives $\ell_d(r, |v|)$ for the pair terms. The two orientations of each pair give the factor $1/2$. Only finitely many displacement vectors contribute, which proves (5). The union contains the disjoint unit cores, so $N_r(L) \geq v_d$.

Slicing perpendicular to the line between the centers gives (2). For $d \geq 2$, its derivative in s is

$$\partial_s \ell_d(r, s) = -v_{d-1}(r^2 - s^2/4)_+^{(d-1)/2}, \quad (7)$$

where $x_+ = \max\{x, 0\}$. This derivative is continuous even at $s = 2r$. Integrating its bound for $s \geq 2$ proves (6), also when $s \geq 2r$. $\square$

The important feature of (6) is that its coefficient tends to zero as $r \downarrow 1$. This is where $d \geq 2$ enters.

3 A uniform neighborhood of a hard optimum

Let L_0 be a local covolume minimum among admissible lattices. It has shortest vector length two, since otherwise a slight uniform contraction would be admissible. Denote its set of length-two vectors by S , and let $\mu > 4$ be the squared length of its next shell. Modulo orthogonal transformations, a nearby lattice has the form $(I + H)^{1/2}L_0$, with H symmetric and small. Put $h = \|H\|_F$, the Frobenius norm. The minimal-vector constraints define the cone

$$\mathcal{C} = \{H = H^T : v^T H v \geq 0 \text{ for every } v \in S\}. \quad (8)$$

For small h , these are the only active admissibility constraints: if $w \notin S \cup \{0\}$, then

$$|(I + H)^{1/2}w|^2 \geq (1 - h)|w|^2 \geq (1 - h)\mu > 4. \quad (9)$$

Lemma 3.1. *There are $a > 0$ and $h_0 > 0$, depending on L_0 , such that*

$$J(H) := \sqrt{\det(I + H)} \geq 1 + ah \quad (H \in \mathcal{C}, h \leq h_0). \quad (10)$$

Proof. Every nonzero $H \in \mathcal{C}$ has $\text{tr } H > 0$. Indeed, if $\text{tr } H \leq 0$, then (9) makes $(I + tH)^{1/2}L_0$ admissible for all sufficiently small $t > 0$. Writing the eigenvalues of H as a_i , however, gives

$$\log \det(I + tH) = \sum_i \log(1 + ta_i) < t \sum_i a_i \leq 0.$$

The inequality is strict because $H \neq 0$. This contradicts the local covolume minimality of L_0 .

The nonempty compact set $\mathcal{C} \cap \{\|H\|_F = 1\}$ therefore has a positive minimum c of $\text{tr } H$. Homogeneity gives $\text{tr } H \geq ch$ throughout the cone. Uniform Taylor expansion yields $J(H) = 1 + \frac{1}{2} \text{tr } H + O(h^2)$. After reducing h_0 , the error is at most $ch/4$, so (10) holds with $a = c/4$. $\square$

Proposition 3.2. *There are a neighborhood of L_0 , modulo rotations, and $r_1 > 1$ such that L_0 strictly maximizes ρ_{r-1} within that neighborhood for every $1 < r < r_1$. The neighborhood is independent of r .*

Proof. Reduce h_0 in Lemma 3.1 and choose $r_0 > 1$ such that

$$r_0 < 2/\sqrt{3}, \quad (1 - h_0)\mu > 4r_0^2.$$

For $h \leq h_0$ and $1 < r \leq r_0$, only vectors from S contribute to the overlap sum, both for L_0 and for AL_0 , where $A = (I + H)^{1/2}$. For $v \in S$, admissibility gives

$$0 \leq |Av| - 2 = \frac{v^T H v}{|Av| + 2} \leq h.$$

Writing $\tau = |S|$, Lemma 2.1 implies

$$N_r(AL_0) \leq N_r(L_0) + b(r)h, \quad b(r) = \frac{\tau}{2}v_{d-1}(r^2 - 1)^{(d-1)/2}. \quad (11)$$

As $d \geq 2$, choose $r_1 \in (1, r_0]$ so that $b(r) < av_d/2$ for $1 < r < r_1$. Put $N_0 = N_r(L_0) \geq v_d$. For $0 < h \leq h_0$,

$$\rho_{r-1}(AL_0) \leq \frac{N_0 + b(r)h}{V(L_0)(1 + ah)} < \frac{N_0}{V(L_0)}.$$

In fact the density loss is at least

$$\frac{(av_d - b(r))h}{V(L_0)(1 + ah)}. \quad (12)$$

The same h_0 works for the whole parameter interval. $\square$

4 Compactness and global selection

We use the classical Mahler compactness theorem in the following form: a sequence of full-rank lattices with a positive common lower bound on shortest-vector length and a finite upper bound on covolume has a subsequence with convergent bases and a full-rank lattice limit. This is the form stated in [5, Definition 2.10 and Theorem 2.11]; that source credits Mahler's original theorem.

First, $v_d \leq V_d \leq 2^d$. The lower bound follows from hard packing density at most one, and $2\mathbb{Z}^d$ supplies the upper bound. Mahler compactness applied to a minimizing sequence shows that V_d is attained. Limits stay admissible and covolume is continuous under basis convergence. Every hard optimum is locally isolated, modulo rotations, by Lemma 3.1. There can be only finitely many inequivalent hard optima: otherwise a sequence of distinct classes would have a basis-convergent subsequence to a hard optimum, contradicting its local isolation. This proves the assertions about $\mathcal{H}_d$ used in the statement.

Proof of Theorem 1.1. For every admissible lattice and $r \geq 1$, inclusion of unit cores and subadditivity of volume give

$$\frac{v_d}{V(L)} \leq \rho_{r-1}(L) \leq \frac{v_d r^d}{V(L)}. \quad (13)$$

Choose the fixed neighborhoods from Proposition 3.2 around the finitely many hard-optimal classes, and take a common small radius interval for all of them.

We claim that, for all sufficiently small $r - 1 > 0$, a lattice matching or beating the best hard optimum in soft density must lie in the union of these neighborhoods. If not, there are $r_n \downarrow 1$ and admissible lattices L_n outside the union with soft densities at least the best hard-optimal soft density. By (13),

$$V_d \leq V(L_n) \leq r_n^d V_d. \quad (14)$$

Mahler compactness gives convergent bases $B_n \rightarrow B_\infty$ for a subsequence, whose limit lattice L_∞ is admissible and has covolume V_d . The matrices $A_n = B_n B_\infty^{-1}$ tend to I . Polar decomposition removes an orthogonal factor, leaving Gram perturbations $H_n = A_n^T A_n - I \rightarrow 0$. Hence L_n eventually belongs to the chosen neighborhood of L_∞ , a contradiction.

Inside such a neighborhood, every lattice other than the central hard optimum, up to rotations, is strictly worse than that center by Proposition 3.2. It is therefore strictly worse than the best hard optimum. Consequently all global soft maximizers are hard optima, and a best hard optimum is a global soft maximizer. This proves attainment rather than presupposing it.

Finally, reduce the common radius interval so that no nonminimal vector of any hard optimum contributes to the overlap sum. Their covolumes all equal V_d , so

$$\rho_{r-1}(L) = \frac{v_d r^d - \frac{1}{2}\tau(L)\ell_d(r, 2)}{V_d} \quad (L \in \mathcal{H}_d).$$

Since $\ell_d(r, 2) > 0$ for $r > 1$, exactly the smallest kissing number maximizes this expression. This proves the theorem. $\square$

Remark 4.1. The sign of the contact selection matters. At fixed covolume, maximizing union volume favors fewer contacts. The finite-cluster minimum-union problem in [2, Theorem 1] instead favors more contacts. The truncated-density conjecture in [4, Conjecture 1.15] is likewise not the assertion proved here.

5 The FCC consequence and a local certificate

Gauss's classical lattice theorem says that the hard optimum in dimension three is uniquely FCC, with density $\pi/(3\sqrt{2})$; see [6, Section 1.6, equation (9)] for an explicit statement including uniqueness. Normalize it as

$$L_F = \sqrt{2}D_3, \quad D_3 = \{z \in \mathbb{Z}^3 : z_1 + z_2 + z_3 \in 2\mathbb{Z}\}.$$

It has covolume $4\sqrt{2}$, twelve minimal vectors $\sqrt{2}(\pm e_i \pm e_j)$, and next squared shell length eight. The three-dimensional lens volume is

$$\ell_3(r, s) = \frac{\pi}{12}(4r + s)(2r - s)^2 \quad (0 \leq s \leq 2r),$$

and thus $\ell_3(1 + \lambda, 2) = 2\pi\lambda^2 + (4\pi/3)\lambda^3$. Substitution into (3), with $\tau = 12$, proves Corollary 1.2 and (4).

Here is a quantitative version of the local step. The FCC cone constraints for a symmetric $H = (h_{ij})$ are

$$h_{ii} + h_{jj} \geq 2|h_{ij}| \quad (i < j).$$

Write $P = h_{11} + h_{22}$, $Q = h_{11} + h_{33}$ and $R = h_{22} + h_{33}$. These numbers are nonnegative, $\text{tr } H = (P + Q + R)/2$, and every matrix entry has absolute value at most $\text{tr } H$. Therefore $\text{tr } H \geq \|H\|_F/3$. For $h = \|H\|_F \leq 1/16$, expansion in the eigenvalues gives

$$\det(I + H) \geq 1 + \frac{1}{3}h - h^2 - h^3 \geq 1 + \frac{1}{4}h, \quad \sqrt{\det(I + H)} \geq 1 + \frac{1}{12}h.$$

In the first inequality, the absolute values of the quadratic and cubic elementary symmetric terms are bounded by h^2 and h^3 . For the last inequality, compare $(1 + h/12)^2$ with $1 + h/4$.

Take $0 < \lambda \leq 1/200$ and $r = 1 + \lambda$. Nonminimal vectors stay outside pair-overlap range, since

$$(1 - 1/16)8 > 4(201/200)^2,$$

and there are no triple intersections. The numerator gain is at most $6\pi(r^2 - 1)h$, whereas $N_r(L_F) \geq 4\pi/3$ and

$$6(r^2 - 1) \leq 6 \cdot \frac{401}{40000} < \frac{1}{9}.$$

The ratio comparison therefore proves strict local maximality in the same $h \leq 1/16$ neighborhood for every $0 < \lambda \leq 1/200$. This is a local certificate only. The global threshold comes from the separate compactness argument and is not claimed to equal $1/200$.

6 Scope, reproducibility, and status

Dimension one is genuinely excluded: the lattice $2(1+\lambda)\mathbb{Z}$ has smaller hard density than $2\mathbb{Z}$ but its radius- $(1+\lambda)$ intervals tile the line, attaining soft density one. Cubical norm balls give the analogous obstruction in every dimension. The vanishing Euclidean lens slope is essential. Arbitrary multi-center periodic configurations do not have the Bravais-lattice Gram cone used above; this proof makes no claim of FCC optimality among general packings.

The accompanying standard-library Python checker uses exact rational arithmetic. Normal and optimized runs each passed 18,218 checks, including 1,701 rational FCC-cone matrices and 2,112 lens-distance cases in odd dimensions three through nine. They also check the FCC polynomial, shell bounds and explicit local constants, and include controls for the excluded hypotheses and pair-counting convention. The byte-identical reports record their finite domains. These are reproducible regression checks of calculations, not a substitute for the general proof or a formal verification of it.

The source and novelty review is bounded to the literature inspected through 9 October 2026; no absolute priority claim is made. The exact conjecture component, earlier local result and classical inputs are distinguished above. AI tools assisted with literature retrieval, mathematical exploration, drafting and computational checks. The manuscript underwent the originating researcher’s detailed self-audit, but has not been independently refereed or verified in a proof assistant.

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