

Soft Density Profiles Do Not Determine Vacant Percolation

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Abstract

For any packing of unit balls in three-dimensional Euclidean space, we construct two fixed modifications preserving its upper and lower soft-density profiles at every fixed outer radius. One modification has an unbounded vacant component at every finite radius. The other has infinitely many nonempty bounded vacant components, and no unbounded one, at every radius at least $2/\sqrt{3}$. Below this radius the vacancy of every unit-core packing is path connected and unbounded. The constructions use a widening empty corridor and lacunary triangulated octahedral barriers, whose fixed-width neighborhoods have volume $O(R^2)$ in an observation ball of radius R . An elementary attainment argument then gives percolating and nonpercolating unrestricted upper-density optimizers at every radius at or above the threshold. No periodicity, saturation or stationarity is imposed.

1 Density and connectivity

Problem 1.4 in the AIM list on soft packings [1] asks about connected interstices and infinite vacant components of optimal soft packings as the softness parameter varies. Before imposing geometric regularity on an optimizer, the meaning of optimality matters. We use the unrestricted upper soft density of Bezdek and Lángi [2, Definition 1]: unit hard cores do not overlap, and density is the asymptotic fraction covered by the union of their expanded closed balls.

The distinction between asymptotic density and local efficiency is classical. Knill [5, Section 2] discusses density-preserving finite deletions. Fejes Tóth, Kuperberg and Kuperberg [3] introduce saturation conditions to distinguish local efficiency from global density, and Kuperberg [6] develops stronger notions of denseness. The triangular throat clearance $2/\sqrt{3} - 1$ in close-packed spheres is also classical; it appears already in Hudson’s abstract [4]. We do not claim these principles or this numerical radius as new.

Our purpose is to give explicit modifications preserving the *whole* soft-density profile of an arbitrary packing, yet having opposite vacant-percolation behavior at every fixed radius above the triangular throat. No density-maximizing configuration need be known.

A *unit-core packing* is a set $C \subset \mathbb{R}^3$ with $|c - c'| \geq 2$ for distinct centers. It is closed and locally finite. Write $B(x, r)$ for an open Euclidean ball, $B_R = B(0, R)$, and put

$$U_r(C) = \bigcup_{c \in C} \overline{B}(c, r), \quad V_r(C) = \mathbb{R}^3 \setminus U_r(C), \quad f_C(r, R) = \frac{\text{vol}(U_r(C) \cap B_R)}{\text{vol}(B_R)}.$$

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The occupied union is closed by local finiteness. The lower and upper soft densities are $\underline{\delta}_C(r) = \liminf_{R \rightarrow \infty} f_C(r, R)$ and $\bar{\delta}_C(r) = \limsup_{R \rightarrow \infty} f_C(r, R)$. The usual softness is $\lambda = r - 1$, with $r \geq 1$. Vacant percolation means existence of an *unbounded* connected component of $V_r(C)$. Since vacancy is open, its connected components are path connected.

Theorem 1. *Put $a = 2/\sqrt{3}$.*

1. *For every unit-core packing C and $0 \leq r < a$, the vacancy $V_r(C)$ is path connected and unbounded.*
2. *Every such C admits unit-core packings C_+ and C_- , independent of r , such that for every fixed finite $r \geq 0$,*

$$|f_{C_\pm}(r, R) - f_C(r, R)| = O_r(R^{-1}). \quad (1)$$

Their entire upper and lower soft-density profiles agree.

3. *$V_r(C_+)$ has an unbounded component for every finite $r \geq 0$.*
4. *For every $r \geq a$, $V_r(C_-)$ has infinitely many nonempty components, all bounded.*

The constants in (1) need not be uniform as r increases. The modified packings need not be periodic or saturated; their empty regions have arbitrarily large inradii. These features are essential to the construction.

2 Negligible modifications and an empty corridor

For $E \subset \mathbb{R}^3$, let $E^{+w} = \{x : \text{dist}(x, E) \leq w\}$.

Lemma 2. *If $C \triangle D \subset E$, then*

$$U_r(C) \triangle U_r(D) \subset E^{+r}, \quad |f_C(r, R) - f_D(r, R)| \leq \frac{\text{vol}(E^{+r} \cap B_R)}{(4\pi/3)R^3}.$$

Thus fixed-neighborhood volumes $O_w(R^2)$ preserve all fixed-radius upper and lower soft densities.

Proof. A point covered by one union and not the other is covered by a ball whose center belongs to the corresponding one-sided difference of center sets. Taking volumes proves the assertion, even when expanded balls overlap. $\square$

Write $x = (t, y) \in \mathbb{R} \times \mathbb{R}^2$, and define

$$E_+ = \{(t, y) : |y| \leq \sqrt{1 + |t|}\}, \quad C_+ = C \setminus E_+.$$

Deletion preserves separation. A point $x \in E_+^{+w} \cap B_R$ satisfies $|x_1| \leq R$ and $|(x_2, x_3)| \leq \sqrt{1 + R + w} + w$. Hence

$$\text{vol}(E_+^{+w} \cap B_R) \leq 2\pi R(\sqrt{1 + R + w} + w)^2 = O_w(R^2),$$

which proves (1) for C_+ .

Fix r . If a retained center $c = (t, y)$ were within distance r of $(T, 0, 0)$, then $|y| \leq r$ and $t \geq T - r$. For $T \geq r^2 + r + 1$, this gives $\sqrt{1 + |t|} > r \geq |y|$, a contradiction. Local finiteness therefore implies $\text{dist}((T, 0, 0), C_+) > r$, with distance to the empty set taken as infinity. The entire tail ray lies in $V_r(C_+)$.

3 Lacunary occupied barriers

Use the minimum-distance-two FCC lattice

$$L = \sqrt{2}\{z \in \mathbb{Z}^3 : z_1 + z_2 + z_3 \equiv 0 \pmod{2}\}.$$

A nonzero even-sum integer vector has squared norm at least two. For even positive N , put

$$\rho = \sqrt{2}N, \quad \Gamma_N = \{x : |x|_1 = \rho\}, \quad S_N = L \cap \Gamma_N, \quad |x|_1 = |x_1| + |x_2| + |x_3|.$$

Lemma 3. *The octahedral surface Γ_N is contained in $U_a(S_N)$.*

Proof. Choose signs on a face and divide coordinates by $\sqrt{2}$. The face becomes $\{z_i \geq 0 : z_1 + z_2 + z_3 = N\}$, whose integer grid divides it into N^2 equilateral triangles of side $\sqrt{2}$. After rescaling, they have side two. Their vertices belong to L , since signs do not change parity and N is even.

A closed equilateral triangle of side two is covered by closed radius- a balls at its vertices. Place its circumcenter at zero, so $|v_i|^2 = a^2$. If $x = \sum_i \alpha_i v_i$, $\alpha_i \geq 0$, $\sum_i \alpha_i = 1$, then

$$\min_i |x - v_i|^2 \leq \sum_i \alpha_i |x - v_i|^2 = a^2 - |x|^2 \leq a^2.$$

Apply this to each small face triangle. Closed occupied balls include the face circumcenters at equality. $\square$

Set

$$N_j = 2^{j+4}, \quad \rho_j = \sqrt{2}N_j, \quad \Gamma = \bigcup_{j \geq 0} \Gamma_{N_j}, \quad S = \bigcup_{j \geq 0} S_{N_j}.$$

Both unions are locally finite.

Lemma 4. *For every fixed $w \geq 0$, $\text{vol}(\Gamma^{+w} \cap B_R) = O_w((R+1)^2)$.*

Proof. Let $b = \sqrt{3}w$. The w -neighborhood of shell j is contained in $\max(\rho_j - b, 0) \leq |x|_1 \leq \rho_j + b$. The one-norm ball of radius s has volume $4s^3/3$, and

$$(\rho + b)^3 - \max(\rho - b, 0)^3 \leq 6b\rho^2 + 2b^3.$$

For $\rho \geq b$ this is equality; for $\rho < b$, the difference between right and left sides is $(b - \rho)^3$. A shell neighborhood can meet B_R only if $\rho_j \leq \sqrt{3}(R + w)$. If m is the largest such index, then

$$\sum_{j=0}^m \rho_j^2 \leq \frac{4}{3}\rho_m^2, \quad m+1 \leq \frac{1}{\rho_0^2} \sum_{j=0}^m \rho_j^2.$$

Consequently

$$\text{vol}(\Gamma^{+w} \cap B_R) \leq \left(8b + \frac{8b^3}{3\rho_0^2}\right) \sum_{j=0}^m \rho_j^2 = O_w((R+1)^2).$$

If there is no such shell, the volume is zero. The cutoff includes shells just outside the observation ball. $\square$

4 Nonempty rooms between the barriers

Put

$$q_j = (3\rho_j/2, 0, 0), \quad h_j = \sqrt{N_j}, \quad H = \bigcup_{j \geq 0} B(q_j, h_j).$$

The distances from q_j to its inner and outer shells are $\rho_j/2$ and $\rho_j/(2\sqrt{3})$, respectively. The latter follows by orthogonal projection to an adjacent face, with nonnegative projected coordinates. All other shells are farther away. Since $N_j \geq 16$,

$$h_j \leq \rho_j/4 < \rho_j/(2\sqrt{3}).$$

Thus the room balls lie strictly between successive shells and miss S . If $B(q_j, h_j + w)$ meets B_R , then

$$\frac{3\rho_j}{2} \leq R + h_j + w \implies \rho_j \leq \frac{4}{5}(R + w).$$

Because $h_j = 2^{-1/4}\rho_j^{1/2}$, geometric summation gives

$$\text{vol}(H^{+w} \cap B_R) \leq \frac{4\pi}{3} \sum_{\rho_j \leq (4/5)(R+w)} (h_j + w)^3 = O_w((R+1)^{3/2}). \quad (2)$$

For example, use $(u+v)^3 \leq 4(u^3 + v^3)$ and $\sum_{j=0}^m \rho_j^{3/2} \leq \rho_m^{3/2}/(1 - 2^{-3/2})$; the number of terms is also bounded by their sum divided by $\rho_0^{3/2}$.

Now define

$$C_- = (C \setminus (H \cup \{x : \text{dist}(x, S) < 2\})) \cup S.$$

Old centers retain their separation, retained old centers are at distance at least two from new ones, and new centers all belong to L . Thus C_- is a unit-core packing. Since $C \triangle C_- \subset H \cup \Gamma^{+2}$, Lemma 4, (2) and Lemma 2 prove (1) for C_- .

All remaining old and new centers are at distance at least h_j from q_j . For fixed r and sufficiently large j ,

$$B(q_j, h_j - r) \subset V_r(C_-). \quad (3)$$

If $r \geq a$, Lemma 3 makes every Γ_{N_j} occupied. The continuous map $x \mapsto |x|_1$ maps a connected vacant set into an interval avoiding every ρ_j , hence into $[0, \rho_0)$ or some (ρ_j, ρ_{j+1}) . Each corresponding region of space is bounded. The balls in (3) lie in different regions, proving infinitely many nonempty components, all bounded, including at the critical radius.

5 Universal connectivity below the throat

We prove the first assertion without a duality argument for an infinite union. For $0 < s < a$, three closed s -balls cannot have a common point: if their centers have centroid $\bar{c}$, then

$$\sum_{i=1}^3 |c_i - x|^2 = 3|\bar{c} - x|^2 + \frac{1}{3} \sum_{i < j} |c_i - c_j|^2 \geq 4,$$

whereas a common point would give a sum at most $3s^2 < 4$.

On the boundary sphere of one s -ball, its intersections with the other closed s -balls are finitely many pairwise disjoint closed spherical caps, possibly singletons. Finiteness follows from local

finiteness, and disjointness from the absence of triple intersections. Each nontrivial cap is smaller than a hemisphere, since its angular radius satisfies $\cos \theta = |c - c'|/(2s) > 0$.

The sphere minus the relative interiors of these caps is path connected. Join two points by piecewise great-circle arcs, and replace every portion inside a cap by an arc on its boundary circle. Only finitely many portions are needed after deleting redundant crossings; an arc already following a boundary requires no replacement. Disjoint cap closures prevent interference. Singleton caps need no removal. The surviving boundary lies outside the interiors of all the balls.

Take $x, y \in V_r(C)$ with $r < a$, and choose

$$r < s < \min\{a, \text{dist}(x, C), \text{dist}(y, C)\}.$$

Only finitely many closed s -balls meet $[x, y]$. The components of their intersection with the segment are finitely many closed intervals, possibly degenerate. In each nondegenerate interval, a finite chain of overlapping ball intervals connects the entrance to the exit. Successive balls have intersecting boundary spheres. Choose a point on each such intersection; it lies in no third ball. Within each ball's surviving boundary, connect successive chosen points, treating entrance and exit similarly. The cap argument gives these arcs while avoiding interiors of every ball, including neighbors that did not meet the original segment. Tangencies and degenerate intervals do not obstruct the construction.

Replacing the occupied segment intervals by these arcs gives a path at distance at least $s > r$ from all centers, hence in the open vacancy. To see unboundedness, if C is bounded it is finite. Otherwise choose centers escaping to infinity. On each associated boundary s -sphere, finitely many pairwise disjoint proper caps cannot cover the sphere. A surviving point has distance at least $s > r$ from every center, and these points escape to infinity. This completes the proof of Theorem 1.

6 Consequences for density optimizers

Lemma 5. *For fixed $r \geq 0$, the value $\Delta(r) = \sup_C \bar{\delta}_C(r)$, over all unit-core packings, is attained.*

Proof. For $k \geq 2$, choose C_k with upper density greater than $\Delta(r) - 1/k$. Starting with $T_1 = 0$, choose T_k so that

$$f_{C_k}(r, T_k) > \Delta(r) - 2/k, \quad T_k \geq k(T_{k-1} + r + 4).$$

The limsup definition allows arbitrarily large such radii. Define

$$D = \bigcup_{k \geq 2} \{c \in C_k : T_{k-1} + 2 \leq |c| \leq T_k - 2\}.$$

Distinct annular pieces have center-norm gaps at least four, so D is a packing. Every point covered by C_k with $T_{k-1} + r + 2 < |x| < T_k - r - 2$ is covered by a retained center. Hence

$$f_D(r, T_k) \geq f_{C_k}(r, T_k) - \left(\frac{T_{k-1} + r + 2}{T_k}\right)^3 - \left[1 - \left(1 - \frac{r + 2}{T_k}\right)^3\right].$$

The two losses are at most k^{-3} and $3/k$. It follows that $\bar{\delta}_D(r) \geq \Delta(r)$, and the reverse inequality holds by definition. $\square$

Corollary 6. *For $r < a$, every unrestricted upper-density optimizer has path-connected unbounded vacancy. For every $r \geq a$, there are both percolating and nonpercolating unrestricted upper-density optimizers; the latter can have infinitely many nonempty vacant components.*

Proof. Apply Theorem 1 to an optimizer supplied by Lemma 5. Its optimal value is preserved. $\square$

The universal path-connectivity guarantee therefore has sharp softness threshold $2/\sqrt{3} - 1$ in this unrestricted three-dimensional setting. No finite radius forces every upper-density optimizer to lose vacant percolation. This does not assert that some optimizer has fully path-connected vacancy above the threshold, and does not settle periodic, saturated, stationary or higher-dimensional versions of the AIM question, or the FCC/BCC optimality problem.

The construction preserves densities in every *fixed* translated observation ball as well, by the same bound inside a larger ball centered at the origin. We claim neither uniformity over moving centers nor preservation of a stationary law or upper Banach density.

The accompanying exact-arithmetic program checks finite shell triangulations, geometric identities and boundary inequalities in specified finite ranges. These tests supplement, not replace, the general proofs. This unrefereed manuscript was developed with AI assistance for proof exploration, literature searches, symbolic checks and drafting. No independent peer review, proof-assistant verification or absolute priority is claimed.

References

- [1] American Institute of Mathematics, *Soft Packings*, Problem 1.4, *Percolation*, <http://aimpl.org/softpack/1/>, accessed 9 October 2026.
- [2] K. Bezdek and Z. Lángi, *Remarks on soft ball packings in dimensions 2 and 3*, arXiv:2310.04574v2 (2024), <https://arxiv.org/abs/2310.04574v2>.
- [3] G. Fejes Tóth, G. Kuperberg and W. Kuperberg, *Highly saturated packings and reduced coverings*, Monatsh. Math. **125** (1998), 127–145, <https://arxiv.org/abs/math/9511225>.
- [4] D. R. Hudson, *Density and Packing in an Aggregate of Mixed Spheres*, J. Appl. Phys. **20** (1949), 154–162, <https://doi.org/10.1063/1.1698327>.
- [5] O. Knill, *Maximizing the packing density on a class of almost periodic sphere packings*, Expositiones Mathematicae **14** (1996), 227–246, <https://people.math.harvard.edu/~knill/oldinterests/kepler/kepler.html>.
- [6] G. Kuperberg, *Notions of denseness*, Geometry & Topology **4** (2000), 277–292, <https://doi.org/10.2140/gt.2000.4.277>.