

Saturated Aperiodic Packings with Finite Stability

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Abstract

We give explicit one-parameter families of saturated unit-ball packings in $\mathbb{R}^2$ and $\mathbb{R}^3$ with trivial translational stabilizer and exact square or cubic grid contact graphs. Every finite subfamily is locally isolated when its complement is fixed. For rational parameters, all centers and the planar equilibrium stresses are rational. The planar proof uses a positive reciprocal stress and a finite-support rigidity argument; a layered extension gives the three-dimensional result. Nevertheless, varying the parameter produces a nontrivial global motion that is continuous in the uniform displacement norm and preserves every contact. The construction thus concerns finite stability, not rigidity against arbitrary motions of infinitely many balls. In particular, it supplies a precisely scoped construction related to Musin’s aperiodic-jamming question without asserting an unrestricted or strict-jamming solution.

1 Introduction and scope

A unit-ball packing is a collection of closed balls of radius one with disjoint interiors. Its center set therefore has pairwise distances at least two. We call it saturated if no further unit ball can be inserted. This means maximality under inclusion, not maximal packing density. Aperiodicity here means that the only translation preserving the center set is the zero translation.

There are several inequivalent notions of stability for an infinite packing. We use the following standard finite-support notion [2, p. 112].

Definition 1. A packing with centers $(p_v)_{v \in V}$ is finitely stable if, for every finite $S \subset V$, its configuration is locally isolated among packings obtained by changing only $(p_v)_{v \in S}$. The isolation neighborhood may depend on S .

Equivalently, no nonconstant continuous motion of a fixed finite set of balls is possible while all other centers remain fixed. We will prove local isolation directly, rather than appeal to this equivalence.

Musin’s Problem 1.35 in the AIM Soft Packings list asks for aperiodic jammed packings in the plane and, in three dimensions, a saturated jammed maximal packing aperiodic in all directions [1]. That entry does not specify the intended notion of jamming. Our result treats finite stability explicitly. It should not be read as a resolution under stronger unrestricted or strict-jamming definitions.

Theorem 2. *For each $0 < \varepsilon \leq 1/10$ there are explicitly specified center sets $P_2(\varepsilon) \subset \mathbb{R}^2$ and $P_3(\varepsilon) \subset \mathbb{R}^3$ such that:*

(i) the unit balls centered there form saturated packings;

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- (ii) the contact graph is exactly the grid on $\mathbb{Z}^2$ or $\mathbb{Z}^3$;
- (iii) the packing is finitely stable;
- (iv) its translational stabilizer is trivial;
- (v) all centers are rational when ε is rational.

Their densities are $\pi/4$ and $\pi/6$, respectively. Varying ε gives a nontrivial global flex, continuous in the uniform displacement norm, preserving every grid contact.

The use of a reciprocal stress and a rigid ordering is classical: Connelly explains this combination for regular circle packings [3, Section IV]. Bezdek, Bezdek and Connelly develop finite and uniform stability and their relation to strut frameworks [2]. Periodic and strict jamming impose different constraints [4]. The contribution here is the explicit rational, nonperiodic family with its positive stress and layered extension, not the general stress criterion or an absolute priority assertion.

A locally jammed grid deformation in the frozen research record AIM-GEOMETRY-0110 uses oscillating cross-shifts [5]. The construction below uses positive summable increments instead. This choice permits an explicit positive equilibrium stress. The spatial construction is a layered extension, not a cyclic three-coordinate version of the planar formula.

2 Rational contact increments

For $n \in \mathbb{Z}$ define

$$U_n = \frac{n}{1 + |n|}, \quad h_n = U_{n+1} - U_n, \quad t_n = \frac{\varepsilon}{4} h_n. \quad (1)$$

Set

$$\ell_n = \frac{2(1 - t_n^2)}{1 + t_n^2}, \quad \delta_n = \frac{4t_n}{1 + t_n^2}, \quad r_n = \frac{\delta_n}{\ell_n} = \frac{2t_n}{1 - t_n^2}. \quad (2)$$

Finally, let $x_0 = s_0 = 0$ and prescribe, for every integer n ,

$$x_{n+1} - x_n = \ell_n, \quad s_{n+1} - s_n = \delta_n. \quad (3)$$

These recurrences define every coordinate by a finite sum, including negative indices.

Lemma 3. Write $c = 199/100$. For $0 < \varepsilon \leq 1/10$, the sequences above satisfy

$$\ell_n^2 + \delta_n^2 = 4, \quad c < \ell_n < 2, \quad 0 < \delta_n \leq \varepsilon h_n \leq \varepsilon/2, \quad |s_n| \leq \varepsilon, \quad (4)$$

$$0 < r_n < \varepsilon, \quad |r_i - r_{i-1}| \leq 2\varepsilon(U_i - U_{i-1}). \quad (5)$$

Moreover, $\delta_n > 0$ tends to zero as $|n| \rightarrow \infty$.

Proof. Direct computation gives

$$h_n = \begin{cases} ((n+1)(n+2))^{-1}, & n \geq 0, \\ ((-n)(1-n))^{-1}, & n \leq -1. \end{cases}$$

Thus $0 < h_n \leq 1/2$, $h_{-1} = h_0 = 1/2$, and the sums over $n \geq 0$ and $n < 0$ both equal one. Adjacent ratios lie in $[1/3, 3]$, so

$$|h_i - h_{i-1}| \leq 2h_{i-1} = 2(U_i - U_{i-1}). \quad (6)$$

We have $0 < t_n \leq 1/80$. The formulas in (2) give the contact identity and

$$\ell_n \geq \frac{12798}{6401} > \frac{199}{100}, \quad \delta_n \leq 4t_n = \varepsilon h_n.$$

Summing the last inequality on either half-line proves $|s_n| \leq \varepsilon$. For $f(t) = 2t/(1 - t^2)$,

$$f'(t) = \frac{2(1+t^2)}{(1-t^2)^2} < 3 \quad (0 \leq t \leq 1/80).$$

In particular $r_n \leq 3t_n < \varepsilon$. Combining the derivative bound with (6) gives

$$|r_i - r_{i-1}| \leq 3|t_i - t_{i-1}| \leq \frac{3}{2}\varepsilon(U_i - U_{i-1}),$$

which implies (5). Positivity and decay of δ_n follow from the same formulas. $\square$

3 The planar packing and its stress

Define

$$p_2(i, j) = (x_i + s_j, x_j + s_i), \quad (i, j) \in \mathbb{Z}^2. \quad (7)$$

The two forward neighbor vectors are

$$a_i = (\ell_i, \delta_i) = \ell_i(1, r_i), \quad b_j = (\delta_j, \ell_j) = \ell_j(r_j, 1). \quad (8)$$

Lemma 4. *The centers (7) give a unit-disk packing with exactly the grid contacts. Its covering radius is at most $\sqrt{2}(1 + \varepsilon) < 2$.*

Proof. The vectors in (8) have length two. If both indices of two centers differ, each absolute coordinate difference is at least $c - 2\varepsilon$. Their squared distance is at least $2(c - 2\varepsilon)^2 > 4$. If just one index changes by magnitude at least two, the corresponding coordinate changes by at least $2c > 2$. This exhausts all nonneighbor pairs.

The bi-infinite sequence x_n has gaps at most two. Given $y = (y_1, y_2)$, choose x_i and x_j within one of the respective coordinates. Since $|s_i|, |s_j| \leq \varepsilon$, $\|y - p_2(i, j)\|^2 \leq 2(1 + \varepsilon)^2 < 4$. A new center must be at distance at least two from every existing center, so the strict covering bound proves saturation. $\square$

The following face vectors supply the essential positive stress:

$$g_{ij} = (X_{ij}, Y_{ij}) = \left(\frac{U_i - r_i U_j}{1 - r_i r_j}, \frac{U_j - r_j U_i}{1 - r_i r_j} \right). \quad (9)$$

Equivalently,

$$X_{ij} + r_i Y_{ij} = U_i, \quad r_j X_{ij} + Y_{ij} = U_j. \quad (10)$$

The denominator is positive by Lemma 3. Put

$$A_{ij} = Y_{ij} - Y_{i,j-1}, \quad B_{ij} = X_{ij} - X_{i-1,j}. \quad (11)$$

Lemma 5. *All A_{ij} and B_{ij} are strictly positive, and*

$$\begin{aligned} & A_{ij}(1, r_i) - A_{i-1,j}(1, r_{i-1}) \\ & + B_{ij}(r_j, 1) - B_{i,j-1}(r_{j-1}, 1) = 0. \end{aligned} \quad (12)$$

Consequently the compression weights A_{ij}/ℓ_i on horizontal edges and B_{ij}/ℓ_j on vertical edges are positive equilibrium weights.

Proof. For B_{ij} write $U = U_i$, $U' = U_{i-1}$, $r = r_i$, $r' = r_{i-1}$, $V = U_j$ and $s = r_j$. Its numerator over the positive denominator $(1 - rs)(1 - r's)$ is

$$(U - U')(1 - r's) + (r - r')(sU' - V).$$

Using $|U'|, |V| < 1$, $r', s < \varepsilon$ and (5), this is at least

$$(U - U')\{1 - \varepsilon^2 - 2\varepsilon(1 + \varepsilon)\} = (U - U')(1 - 2\varepsilon - 3\varepsilon^2) > 0.$$

Interchanging the indices proves positivity of A_{ij} .

Equations (10) imply

$$g_{ij} - g_{i,j-1} = A_{ij}(-r_i, 1), \quad g_{ij} - g_{i-1,j} = B_{ij}(1, -r_j).$$

Equality of mixed differences gives

$$A_{ij} - A_{i-1,j} = -r_j B_{ij} + r_{j-1} B_{i,j-1},$$

and

$$B_{ij} - B_{i,j-1} = -r_i A_{ij} + r_{i-1} A_{i-1,j}.$$

These are the two components of (12). Dividing by the corresponding ℓ converts normalized directions to the actual edge vectors (8). $\square$

Our weights are compressive and positive. They are the negatives of the proper strut stresses under the sign convention in [2]. Every vertex has only four incident edges, so equilibrium requires no infinite summability assumption.

4 From the stress to finite stability

A finitely supported velocity field v is admissible to first order if, on every contact edge $\{p, q\}$,

$$(q - p) \cdot (v_q - v_p) \geq 0. \tag{13}$$

Each edge is counted once; the expression is independent of its orientation.

Proposition 6. *Every finitely supported velocity field satisfying (13) on the planar packing is zero.*

Proof. Multiply (13) by the positive weight from Lemma 5 and sum over all edges incident to the finite support. Regrouping by vertices and using equilibrium makes the sum zero. There are only finitely many nonzero summands. Since all weights are positive and all derivatives nonnegative, every contact derivative is zero.

If the support is nonempty, choose (i, j) in it maximizing $i + j$. Its two forward neighbors have zero velocity, whence $a_i \cdot v_{ij} = b_j \cdot v_{ij} = 0$. But

$$\det(a_i, b_j) = \ell_i \ell_j (1 - r_i r_j) > 0.$$

This forces $v_{ij} = 0$, a contradiction. $\square$

Lemma 7. *For a locally finite packing in $\mathbb{R}^d$, the absence of nonzero finitely supported admissible first-order velocities implies finite stability.*

Proof. Fix a finite moving set S . Suppose distinct feasible configurations converge to the original one with all centers outside S fixed. Let $u^{(m)} \in \mathbb{R}^{d|S|}$ be their displacement vectors and put $\eta_m = \|u^{(m)}\| > 0$. A subsequence of $u^{(m)}/\eta_m$ converges to a unit vector v , extended by zero off S .

For an original contact with vector $a = q - p$ and displacement difference $d_m = u_q^{(m)} - u_p^{(m)}$, feasibility says

$$2a \cdot d_m + \|d_m\|^2 \geq 0.$$

Divide by $2\eta_m$ and let $m \rightarrow \infty$. The quadratic term tends to zero, yielding $a \cdot (v_q - v_p) \geq 0$. Thus v is a nonzero finitely supported admissible velocity, a contradiction. The original configuration is locally isolated for each S . $\square$

Proposition 6 and Lemma 7 prove planar finite stability. In particular, the argument concerns actual motions, not merely the outcome of a linear-programming test. The positive stress need not have a positive lower bound independent of the edge, and no uniform isolation radius is asserted.

5 A layered packing in three dimensions

Define

$$p_3(i, j, k) = (x_i + s_j + s_k, x_j + s_i, x_k), \quad (i, j, k) \in \mathbb{Z}^3. \quad (14)$$

The k th horizontal layer is a copy of the planar packing translated by $(s_k, 0, x_k)$. The three forward contact vectors are

$$(\ell_i, \delta_i, 0), \quad (\delta_j, \ell_j, 0), \quad (\delta_k, 0, \ell_k).$$

Proposition 8. *The centers (14) give a saturated, finitely stable unit-ball packing with contact graph exactly the cubic grid.*

Proof. The same-layer contacts are known. Two layers whose indices differ by at least two have vertical separation at least $2c > 2$. For adjacent layers and unequal planar indices, the horizontal distance is at least $2 - \delta_k \geq 2 - \varepsilon/2$, by the planar packing property and the triangle inequality. The vertical separation is $\ell_k > c$. Thus the squared distance is greater than $(2 - \varepsilon/2)^2 + c^2 > 4$. For equal planar indices in adjacent layers the distance is exactly two.

For any point of $\mathbb{R}^3$, choose a height x_k within one of its third coordinate. The planar covering estimate in that layer gives squared distance at most

$$1 + 2(1 + \varepsilon)^2 < 4.$$

This proves saturation.

Let v be a finitely supported admissible velocity. Within each layer, its horizontal components satisfy every planar contact inequality. Proposition 6 makes them zero. The remaining contact inequalities are

$$\ell_k(v_{i,j,k+1}^z - v_{i,j,k}^z) \geq 0.$$

For each fixed (i, j) the vertical velocities form a nondecreasing sequence with finite support, hence vanish identically. Lemma 7 now proves finite stability in $\mathbb{R}^3$. $\square$

6 Aperiodicity, density and the infinite flex

Proposition 9. *Both packings have trivial translational stabilizer.*

Proof. A translation preserving the center set preserves the contact graph and each contact vector. Every forward grid direction is recognized by its unique large positive coordinate $\ell_n > c$; the other coordinates have magnitude at most $\varepsilon/2$. Thus the induced automorphism of the grid preserves each oriented generator. On a connected grid it must be a fixed index shift: $(i, j) \mapsto (i + a, j + b)$, or $(i, j, k) \mapsto (i + a, j + b, k + d)$.

Preservation of contact vectors forces $\delta_{i+a} = \delta_i$ and $\delta_{j+b} = \delta_j$ for all indices; in dimension three it also forces $\delta_{k+d} = \delta_k$. A strictly positive sequence tending to zero at both ends has no nonzero integer period. Therefore $a = b = 0$, and also $d = 0$ in dimension three. The original translation fixes every center and is zero. $\square$

All coordinates are rational when ε is rational, because (1)–(3) use rational operations and finite sums. To compute the densities, observe that

$$2 - \ell_n = \frac{\delta_n^2}{2 + \ell_n}, \quad \sum_{n \in \mathbb{Z}} \delta_n \leq 2\varepsilon, \quad \max_n \delta_n \leq \varepsilon/2.$$

Consequently

$$\sum_{n \in \mathbb{Z}} (2 - \ell_n) \leq \frac{\varepsilon^2}{2 + c}, \quad |x_n - 2n| \leq \frac{\varepsilon^2}{2 + c}. \quad (15)$$

Together with $|s_n| \leq \varepsilon$, this gives a bounded-displacement bijection from $2\mathbb{Z}^d$ to $P_d(\varepsilon)$. The counting discrepancy in a radius- R ball is confined to a shell of bounded width and is $O(R^{d-1})$. The center density is therefore 2^{-d} , giving packing densities $\pi/4$ and $\pi/6$.

Proposition 10. *The parameter family is a nontrivial global flex, continuous in the uniform displacement norm.*

Proof. All grid contacts remain at distance two, and the noncontact bounds hold throughout $0 < \varepsilon \leq 1/10$. For uniform continuity, put $\alpha_n = h_n/4$. Then

$$\delta_n(\varepsilon) = \frac{4\varepsilon\alpha_n}{1 + \varepsilon^2\alpha_n^2}, \quad 2 - \ell_n(\varepsilon) = \frac{4\varepsilon^2\alpha_n^2}{1 + \varepsilon^2\alpha_n^2}.$$

On $[0, 1/10]$ their derivatives have absolute values at most $4\alpha_n = h_n$ and $8\varepsilon\alpha_n^2$, respectively. These bounds are summable in n . Thus all finite cumulative sums defining s_n and $x_n - 2n$ are uniformly Lipschitz in ε , independently of n . Equations (7) and (14) give the required uniform displacement control.

The distance between the labeled planar centers $(0, 0)$ and $(1, 1)$ equals $\sqrt{2}(\ell_0 + \delta_0)$. For $t = t_0 = \varepsilon/8$,

$$\ell_0 + \delta_0 = \frac{2(1 + 2t - t^2)}{1 + t^2}, \quad \frac{d}{dt}(\ell_0 + \delta_0) = \frac{4(1 - 2t - t^2)}{(1 + t^2)^2} > 0.$$

The same pair in the $k = 0$ layer works in dimension three. Hence the labeled motion is not a Euclidean rigid motion. $\square$

These propositions finish Theorem 2. The global flex moves infinitely many balls, whereas finite stability fixes the complement of each finite moving set. There is therefore no contradiction. The global flex does not by itself decide uniform finite-rearrangement stability in the sense of [2]. Strict jamming, statistical disorder and optimal density are not conclusions of this construction.

7 Reproducibility

The accompanying standard-library Python program uses exact rational arithmetic. For four rational parameter values it checks 263652 pairs in finite boxes, 4356 stress vertices, 1028 scalar indices, the summation-by-parts identity on 98 edges, and the uniform numerical margins. Three negative controls detect an out-of-range parameter, a wrong equilibrium sign, and an incorrect rigid-motion claim. Normal and optimized interpreter runs give identical outputs. These finite regressions supplement the all-index proofs above; they are not a substitute for them or a formal verification.

The construction and exposition were developed with AI assistance and checked by mathematical self-audit and the supplied exact computations. This preprint has not undergone independent peer review.

References

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