

Entropy Selection and Shape Drift for Gaussian Lattice Minima

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18 October 2026

Abstract

For each perfect strictly eutactic lattice, we determine the large-parameter Gaussian energy minimum in a fixed neighborhood of its shape. The minimizing shape moves by order t^{-1} , and the leading energy coefficient is the exponential of the maximum entropy of an isotropic probability measure on the shortest vectors. We give an explicit negative correction of order t^{-1} to this coefficient. Uniform localization and a summable tail estimate justify minimization of the expansion. Globally, among unit-covolume lattices, the formula selects Hermite-optimal classes first by the entropy coefficient and then by the squared drift. Poisson summation gives the corresponding small-parameter formula. An exact nine-dimensional cyclic-group example has 160 shortest vectors but optimized coefficient strictly below 160. The results concern Bravais lattices, not arbitrary point configurations; the general Gaussian energy question is not resolved.

1 The lattice problem and main results

For a full-rank Euclidean lattice $L \subset \mathbb{R}^d$, define

$$\theta_L(t) = \sum_{v \in L} e^{-t|v|^2}, \quad \mu(L) = \min_{0 \neq v \in L} |v|^2, \quad t > 0.$$

We use the ordered-pair energy $E_t(L) = \theta_L(t) - 1$. For the unordered-pair convention every energy below is divided by two. All shape variations preserve covolume. Matrices act on the ambient Euclidean space, and shapes are considered up to Euclidean isometry.

The small-parameter Gaussian question in AIM Problem 1.34 [1] motivates the fixed-density lattice problem. Poisson summation exchanges small and large parameters on the dual lattice:

$$\theta_L(\alpha) = \left(\frac{\pi}{\alpha}\right)^{d/2} \theta_{L^*}(\pi^2/\alpha) \quad \text{if } \text{covol}(L) = 1. \quad (1)$$

The shortest length controls the exponential rate at large parameter. However, minimizing the energy is not justified by taking a pointwise shortest-shell limit: the minimizing shape itself can move. We determine that movement and its effect on the leading coefficient.

Let H be a perfect strictly eutactic lattice, put $m = \mu(H)$, and let

$$S = S(H) = \{v \in H : |v|^2 = m\}.$$

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Both signs of every shortest vector belong to S . Perfection means that the matrices vv^T , $v \in S$, span $\text{Sym}(\mathbb{R}^d)$; strict eutaxy means that

$$I = \sum_{v \in S} a_v vv^T \quad \text{for some } a_v > 0.$$

Write $V = \text{Sym}_0(\mathbb{R}^d)$ for the trace-zero symmetric matrices with the Frobenius norm. Define

$$\Phi_H(B) = \sum_{v \in S} e^{-v^T B v}, \quad B \in V. \quad (2)$$

Theorem 1 (Local minimum and entropy). *Suppose $d \geq 2$. The function Φ_H is coercive and strictly convex and has a unique minimizer B_H . Set*

$$c_H = \Phi_H(B_H), \quad D_H = \text{tr}(B_H^2).$$

There is a fixed sufficiently small neighborhood of H in fixed-covolume shape space such that, for all sufficiently large t , $\theta_L(t)$ has a unique minimum over that neighborhood, up to isometry. In an exponential chart its shape is

$$H_t = e^{C_t/2} H, \quad C_t = \frac{B_H}{t} + O(t^{-2}), \quad (3)$$

and its energy is

$$\theta_{H_t}(t) - 1 = e^{-mt} \left(c_H - \frac{m c_H D_H}{2dt} + O(t^{-2}) \right). \quad (4)$$

Moreover,

$$\log c_H = \max \left\{ - \sum_{v \in S} p_v \log p_v : p_v \geq 0, \sum p_v = 1, \sum p_v vv^T = \frac{m}{d} I \right\}, \quad (5)$$

where $0 \log 0 = 0$. In particular $c_H \leq |S|$, with equality if and only if the uniform measure on S is isotropic, or equivalently $B_H = 0$.

The uniqueness assertion concerns the minimum over the selected neighborhood. It is not a classification of all stationary points. The equality condition is strong eutaxy of the first shell; it imposes no design condition on the other shells.

Let $\gamma_d = \max_{\text{covol}(L)=1} \mu(L)$, and let $\mathcal{H}_d$ be the finite set of unit-covolume Hermite-optimal isometry classes. These classes are perfect and strictly eutactic by Voronoi theory [7, Theorems 3–4]. Put

$$c_* = \min_{H \in \mathcal{H}_d} c_H, \quad D_* = \max\{D_H : H \in \mathcal{H}_d, c_H = c_*\}. \quad (6)$$

Theorem 2 (Global lattice selection). *For fixed $d \geq 2$, as $t \rightarrow \infty$,*

$$\min_{\text{covol}(L)=1} (\theta_L(t) - 1) = e^{-\gamma_d t} \left(c_* - \frac{\gamma_d c_* D_*}{2dt} + O(t^{-2}) \right). \quad (7)$$

For all sufficiently large t , every global minimizer is one of the local minimizing branches with $c_H = c_*$ and $D_H = D_*$. Branches tied at both orders are not distinguished by this statement. Equivalently, as $\alpha \rightarrow 0^+$,

$$\begin{aligned} \inf_{\text{covol}(L)=1} E_\alpha(L) &= \left(\frac{\pi}{\alpha} \right)^{d/2} - 1 \\ &\quad + \left(\frac{\pi}{\alpha} \right)^{d/2} e^{-\pi^2 \gamma_d / \alpha} \left(c_* - \frac{\alpha \gamma_d c_* D_*}{2d\pi^2} + O(\alpha^2) \right). \end{aligned} \quad (8)$$

The small-parameter minimizing lattices are the duals of the branches in (3), evaluated at $t = \pi^2 / \alpha$.

These formulas do not determine unknown Hermite constants or unknown Hermite-optimal classes. At density ρ , replace α in (8) by $\alpha\rho^{-2/d}$, since a density- ρ lattice is $\rho^{-1/d}$ times a unit-covolume lattice. In dimension one the shape space is trivial and the formulas hold with $\gamma_1 = 1, c_* = 2, D_* = 0$, directly from the theta series.

Relation to previous work

The qualitative distinction between packing extrema and eventual energy extrema is classical. Ryshkov [5] studied eventual zeta optimality, including localization near densest classes and the role of strong criticality. Sarnak and Strömbergsson [6, Section 5] established theta local minima for the distinguished lattices in dimensions 4, 8, 24. Joharian, Vallentin and Zimmermann [4, Theorem 3.7] treat asymptotic local optimality of fixed extremal even modular lattices under additional hypotheses. These fixed-lattice results and the classical packing theory are not claimed here.

Our contribution is the uniform moving-minimum argument for every perfect strictly eutactic lattice, the resulting entropy coefficient, and the explicit next correction. The maximum-entropy variational identity itself is standard. The cyclic-group lattice used below is also known [2]. The frozen UnsolvedMath analysis associated with the AIM question already contains the elementary shortest-length estimate and pointwise shell limit [8]; our argument addresses the nonuniform minimization step. No absolute priority is asserted.

2 Localization and the moving minimum

Lemma 3 (Sharp packing loss). *There are $\eta > 0$ and $\varepsilon > 0$ such that*

$$\min_{v \in S} v^T B v \leq -\eta \|B\|_F \quad (B \in V)$$

and, whenever $C \in V$ and $\|C\|_F \leq \varepsilon$,

$$\mu(e^{C/2}H) \leq m - \frac{\eta}{2} \|C\|_F. \quad (9)$$

Proof. If $B \neq 0$ and all $v^T B v$ were nonnegative, strict eutaxy would give

$$0 = \text{tr } B = \sum_{v \in S} a_v v^T B v.$$

All these terms would vanish, and perfection would imply $B = 0$. Thus some term is negative. Compactness of the unit sphere in V gives a uniform $\eta > 0$. Uniform Taylor expansion on the finite set S yields

$$\mu(e^{C/2}H) \leq \min_{v \in S} v^T e^C v \leq m + \min_{v \in S} v^T C v + K \|C\|_F^2.$$

Shrinking ε proves (9). □

The same inequality implies $\Phi_H(B) \geq e^{\eta \|B\|_F}$, so Φ_H is coercive. For $A \neq 0$ in V , perfection gives

$$D^2 \Phi_H(B)[A, A] = \sum_{v \in S} e^{-v^T B v} (v^T A v)^2 > 0. \quad (10)$$

This proves existence and uniqueness of B_H .

Lemma 4 (Uniform expansion). *On every fixed bounded ball in V , as $t \rightarrow \infty$,*

$$F_t(B) := e^{mt}(\theta_{e^{B/(2t)}H}(t) - 1) = \Phi_H(B) + \frac{\Psi_H(B)}{t} + O(t^{-2}) \quad (11)$$

in C^2 , where

$$\Psi_H(B) = -\frac{1}{2} \sum_{v \in S} e^{-v^T B v} v^T B^2 v.$$

Proof. Let $\lambda_2 > m$ be the next squared norm in H , and take $\|B\|_F \leq K$. Since $e^{B/t} \succeq (1 - K/t)I$,

$$t v^T e^{B/t} v \geq (t - K)|v|^2.$$

The first two B -derivatives of $t v^T e^{B/t} v$ have norm at most a constant times $|v|^2$, uniformly for $t \geq 1$. This follows from the matrix exponential series, or its integral derivative formula. Therefore every derivative of order at most two of a theta summand, after multiplication by e^{mt} , is bounded by

$$K'(1 + |v|^4)e^{mt - (t-K)|v|^2}.$$

Fix $t_0 > 0$. For $t \geq K + t_0$ and $v \notin S \cup \{0\}$, this is at most

$$K' e^{-(\lambda_2 - m)t + (K + t_0)\lambda_2} (1 + |v|^4) e^{-t_0|v|^2}. \quad (12)$$

The sum of the last factor over the lattice converges. The entire nonminimal tail and its first two derivatives are consequently exponentially small.

On the finite shortest shell,

$$t(v^T e^{B/t} v - m) = v^T B v + \frac{v^T B^2 v}{2t} + O(t^{-2})$$

with a uniform C^2 remainder on the ball. Expanding the scalar exponential and adding the tail proves (11). $\square$

Proof of Theorem 1. Choose a closed small chart ball $\|C\|_F \leq \varepsilon$ on which (9) holds. Its image is compact, and theta is continuous there by locally uniform convergence. Let C_t be a minimizing point on this ball. For fixed $t_0 > 0$ and $t \geq t_0$, comparison with $C = 0$ gives

$$2e^{-t\mu(e^{C_t/2}H)} \leq \theta_{e^{C_t/2}H}(t) - 1 \leq \theta_H(t) - 1 \leq A e^{-mt}, \quad A = e^{mt_0}(\theta_H(t_0) - 1) \geq 2.$$

Combining this with (9), we obtain

$$\|C_t\|_F \leq \frac{2 \log(A/2)}{\eta t}. \quad (13)$$

In particular the minimum is interior for large t , and $B_t = tC_t$ lies in a fixed ball. Enlarge that ball to contain B_H in its interior. By (10), the Hessian of Φ_H has a positive uniform lower bound there. Lemma 4 gives the same property for F_t when t is large. All minimizing points lie in that ball by (13), so strict convexity makes them equal. The finite isometry stabilizer of H acts by conjugation on the chart; uniqueness makes this minimizer invariant. Thus the statement descends to shape space.

At the minimizing point, (11) gives $\nabla \Phi_H(B_t) = O(t^{-1})$. Integrating the positive Hessian on the segment between B_H and B_t shows $B_t - B_H = O(t^{-1})$, proving (3). Since $\nabla \Phi_H(B_H) = 0$, the value expansion is

$$F_t(B_t) = c_H + \frac{\Psi_H(B_H)}{t} + O(t^{-2}).$$

Stationarity on the trace-zero space makes

$$M_H = \sum_{v \in S} e^{-v^T B_H v} v v^T$$

scalar. Its trace is mc_H , hence $M_H = (mc_H/d)I$, and

$$\Psi_H(B_H) = -\frac{1}{2} \text{tr}(B_H^2 M_H) = -\frac{mc_H}{2d} \text{tr}(B_H^2).$$

This proves (4), including its negative sign.

For the entropy identity, put $p_v^* = e^{-v^T B_H v} / c_H$. The formula for M_H makes p^* a feasible positive probability distribution in (5). If p is feasible and $q_v = e^{-v^T B v} / \Phi_H(B)$, nonnegative relative entropy gives

$$\begin{aligned} 0 \leq \sum_{v \in S} p_v \log(p_v / q_v) &= -H(p) + \log \Phi_H(B) + \text{tr} \left(B \sum_{v \in S} p_v v v^T \right) \\ &= -H(p) + \log \Phi_H(B). \end{aligned}$$

Equality holds at $B = B_H, p = p^*$. Thus (5) holds; strict concavity gives uniqueness of the maximizing distribution. Finally $c_H \leq \Phi_H(0) = |S|$, with equality precisely when zero minimizes Φ_H , that is, $\sum_{v \in S} v v^T = (m|S|/d)I$. $\square$

3 Global minimization and Poisson duality

Proof of Theorem 2. For fixed $t > 0$, theta is proper on unit-covolume lattice shape space. Indeed, if $\mu(L) \rightarrow 0$, the multiples of a shortest vector alone give a divergent lower bound $\sum_{k \in \mathbb{Z}} e^{-tk^2 \mu(L)}$. Mahler compactness and continuity therefore give a global minimizer L_t .

Compare L_t with any Hermite-optimal H_0 , as above. For $t \geq t_0$,

$$0 \leq \gamma_d - \mu(L_t) \leq \frac{\log(A_0/2)}{t}, \quad A_0 = e^{\gamma_d t_0} (\theta_{H_0}(t_0) - 1). \quad (14)$$

Hence all minimizers for large t lie in a common Mahler-compact set, and every limit is Hermite-optimal. By Voronoi's theorem these lattices are perfect and strictly eutactic and form finitely many isometry classes [7]. Alternatively, compactness and the isolation supplied by Lemma 3 give the required finiteness.

Every global minimum eventually belongs to the union of the finitely many fixed open neighborhoods from Theorem 1. Otherwise a sequence outside that union would have a Hermite-optimal limit, a contradiction. Inside each neighborhood a global minimum must be the unique neighborhood minimum. The global value is therefore the minimum of finitely many expansions (4), all with $m = \gamma_d$. The least c_H wins first, and among ties the greatest D_H wins because the correction is negative. The finite gaps dominate the uniform $O(t^{-2})$ errors. This proves (7) and the branch-selection assertion. Finally (1) and bijectivity of lattice duality prove (8). $\square$

The lattice restriction is essential. Non-Bravais comparisons can change Gaussian ground states [3]. The preceding proof gives no bound uniform in the number of points of a periodic motif and does not establish crystallization among arbitrary configurations.

4 An exact cyclic-group example

Consider the known Abelian-group lattice [2]

$$L = \left\{ x \in \mathbb{Z}^{10} : \sum_{j=0}^9 x_j = 0, \sum_{j=0}^9 jx_j \equiv 0 \pmod{10} \right\} \subset W = \mathbf{1}^\perp.$$

It has dimension nine and covolume $10\sqrt{10}$: the congruence is the kernel of a surjection $A_9 \rightarrow \mathbb{Z}/10\mathbb{Z}$, and $\text{covol}(A_9) = \sqrt{10}$. All squared norms are even. Norm two would require $e_i - e_j$ with distinct i, j , which violates the congruence. The shortest shell is exactly

$$S = \{e_a + e_b - e_c - e_d : a < b, c < d, |\{a, b, c, d\}| = 4, a + b \equiv c + d \pmod{10}\}. \quad (15)$$

Indeed these vectors have squared norm four; the only other integer norm-four pattern is a single entry of magnitude two, excluded by the zero-sum constraint. There are 160 vectors in (15).

For each $v \in S$, set $x_i = v_i - v_9$, $0 \leq i < 9$, and form the row $(x_i x_j)_{0 \leq i \leq j < 9}$. The accompanying exact certificate lists 45 rows whose determinant is

$$4000000000000000000 \neq 0. \quad (16)$$

Because $e_i - e_9$, $0 \leq i < 9$, is a basis of W , this proves that the shortest-vector projectors span $\text{Sym}(W)$, of dimension 45. Thus L is perfect.

Put $s(r) = (-1)^r$, and let $a(r) = 4$ if $5 \mid r$, and $a(r) = -1$ otherwise. Define matrices, for $0 \leq i, j < 10$, by

$$(P_1)_{ij} = \frac{s(i-j)a(i-j)}{10}, \quad (P_2)_{ij} = \frac{a(i-j)}{10}, \quad (P_5)_{ij} = \frac{s(i-j)}{10}.$$

They are mutually orthogonal projections of ranks 4, 4, 1 and sum to $P_W = I - J/10$. Write $q(v) = (v^T P_1 v, v^T P_2 v, v^T P_5 v)$. The complete shell data are

$q(v)$	Multiplicity	$w(v)$
$(2/5, 2, 8/5)$	20	18/13
$(1, 3, 0)$	40	1
$(2, 2, 0)$	60	1
$(12/5, 0, 8/5)$	20	12/13
$(4, 0, 0)$	20	1

and direct rational summation gives

$$\sum_{v \in S} w(v)vv^T = \frac{960}{13}P_W. \quad (17)$$

All weights are positive, so L is strictly eutactic. This is an explicit certificate for this lattice, not a solution of the general Abelian-group eutaxy question. Its failure of strong eutaxy is consistent with the known classification [2].

For $B_0 = P_1 - P_2$, considered on W , we have $\text{tr } B_0 = 0$ and

$$\sum_{v \in S} v^T B_0 v = 16, \quad \sum_{v \in S} (v^T B_0 v)^2 = \frac{3232}{5}, \quad |v^T B_0 v| \leq 4.$$

Consequently $D\Phi_L(0)[B_0] = -16 < 0$, so $B_L \neq 0$. Taking $\varepsilon = 1/100$, the inequalities $e^x \leq 1 + x + x^2 e^{|x|}/2$ and $e^r \leq (1 - r)^{-1}$ for $0 \leq r < 1$ give

$$\begin{aligned} c_L \leq \Phi_L(\varepsilon B_0) &\leq 160 - 16\varepsilon + \frac{(3232/5)\varepsilon^2}{2(1 - 4\varepsilon)} \\ &= \frac{479621}{3000} < 160. \end{aligned} \tag{18}$$

Thus an actual extreme lattice has optimized Gaussian coefficient strictly below its kissing number. This is a local example; L is not asserted to be Hermite-optimal in dimension nine. If L is rescaled by a factor a , its optimizing matrix is rescaled by a^{-2} and its coefficient c_L is unchanged.

5 Reproducibility

The source package includes a standard-library Python checker and its exact output. Two independent enumerations give the 160 shortest vectors. Rational row reduction gives rank 45; integer Bareiss elimination computes (16), with three modular determinant cross-checks. All entries of the projection and covariance identities, the positive weights, and the rational bound (18) are checked. Normal and optimized Python runs agree on 1,510 checks. These finite certificates support the example, not the general analytic argument of Sections 2 and 3.

This work used AI assistance in research, drafting and exact-checker development. The written proof was self-audited; the preprint is unrefereed and has not been independently reviewed or formally verified.

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