

# Balanced Laurent Potentials for Finite Cluster Actions

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## Abstract

We give counterexamples to the uniqueness assertion for maximum potentials in Theorem 4.7 of Ishibashi's *Fixed point theorem for cluster modular groups*, arXiv:2603.14338v2. Both a rank-one example and a full-rank type  $A_2$  example satisfy its filling conditions and have nontrivial finite modular groups, but their maximum potentials have continua of minimizers. More generally, adjoining a sufficiently large positive integral constant can make any prescribed compact set consist of minimizers. The finite-subgroup fixed-point conclusion is nevertheless preserved: the group-averaged sum of positive Laurent functions with balanced support is coercive and strictly convex in a global logarithmic seed chart. Its unique minimizer is fixed by the group. The logarithm of this sum has the same properties. This gives a replacement proof of the credited fixed-point conclusion using balanced support alone, without the additional slope-span condition. No properness result for infinite cluster modular groups is claimed.

## 1 Setup and the assertion at issue

Let  $M = \mathcal{A}_s(\mathbb{R}_{>0})$  be a positive cluster  $\mathcal{A}$ -manifold. Each seed chart identifies  $M$  globally with  $(\mathbb{R}_{>0})^d$ ; its logarithmic chart identifies  $M$  with  $\mathbb{R}^d$  [1, Definition 3.7]. Fix such a chart, with coordinates  $x = (x_1, \dots, x_d)$  and  $u = \log x$ . A positive Laurent polynomial has the form

$$F(x) = \sum_{\alpha \in S_F} c_{F,\alpha} x^\alpha, \quad 0 < c_{F,\alpha}, \quad S_F \subset \mathbb{Z}^d \text{ finite.}$$

The support  $\text{Supp } F$  is  $S_F$ , and

$$\text{Slope}(F) = \text{span}_{\mathbb{R}}\{\alpha - \beta : \alpha, \beta \in S_F\}.$$

In particular, a constant has zero slope space. A function is *universally positive Laurent* if it has a positive Laurent expression in every seed chart. The source's semiring  $\mathcal{U}^+$  uses positive integral coefficients [1, Definition 3.6]; our examples obey this stronger requirement. Cluster modular transformations preserve  $\mathcal{U}^+$  by pullback.

**Definition 1.1.** A finite collection  $\Lambda$  has *balanced support* in the chosen chart if  $0$  lies in the full-dimensional interior of  $\text{conv}(\bigcup_{F \in \Lambda} \text{Supp } F) \subset \mathbb{R}^d$ .

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This is equivalent to requiring a subset of the support union whose convex hull has 0 in its full-dimensional interior. A filling set in [1, Definition 4.5] is a finite subset  $\Lambda \subset \mathcal{U}^+$  with balanced support for some seed and, for that seed's exchange matrix  $\varepsilon$ , the further conditions

$$\text{Slope}(F) \subset (\ker \varepsilon)^\perp \quad (F \in \Lambda), \quad \text{span}_{\mathbb{R}} \bigcup_{F \in \Lambda} \text{Slope}(F) = (\ker \varepsilon)^\perp. \quad (1)$$

For a finite subgroup  $G$  of the cluster modular group put

$$\Lambda_G = \bigcup_{g \in G} g^* \Lambda, \quad L_{\max}(p) = \max_{F \in \Lambda_G} \log F(p). \quad (2)$$

Theorem 4.7 of [1] asserts that this function has a unique minimizing *point* for every filling set. The next section disproves that assertion as written. Section 3 repairs the argument for the finite-subgroup fixed-point conclusion of its Theorem 4.8. That conclusion is not being refuted or claimed as newly discovered here.

## 2 Flat maximum potentials

**Proposition 2.1.** *For the coefficient-free rank-one cluster structure, there is a filling set invariant under a nontrivial finite modular group whose maximum potential has an interval of minimizers.*

*Proof.* The exchange matrix is  $(0)$ , and the two cluster variables are  $x$  and  $x' = 2/x$ . Mutation defines the order-two map  $x \mapsto 2/x$ . Set

$$\Lambda = \{x, 2/x, 4\}, \quad G = \langle x \mapsto 2/x \rangle.$$

In the other chart these functions are  $2/x'$ ,  $x'$  and 4, so their Laurent expressions have positive integral coefficients in both charts. The support contains  $-1$  and  $1$ , whereas all slope spaces are zero, as required by (1). The set is  $G$ -invariant. But

$$L_{\max}(x) = \log \max\{x, 2/x, 4\}$$

has minimum  $\log 4$  at exactly  $x \in [1/2, 4]$ . □

The use of  $2/x$  is important:  $x^{-1} = x'/2$  would not have integral coefficients in the other chart. The group still has the fixed point  $\sqrt{2}$ , in agreement with the fixed-point conclusion.

**Proposition 2.2.** *The unique-minimizer assertion also fails for an invertible exchange matrix, with a nontrivial finite modular group and a two-dimensional set of minimizers.*

*Proof.* Use the type  $A_2$  exchange matrix  $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Its five cyclically ordered cluster variables are

$$q_1 = x, \quad q_2 = y, \quad q_3 = \frac{1+y}{x}, \quad q_4 = \frac{1+x+y}{xy}, \quad q_5 = \frac{1+x}{y}. \quad (3)$$

They satisfy  $q_i q_{i+2} = 1 + q_{i+1}$  with indices modulo five. Consequently  $T(x, y) = (y, (1+y)/x)$  cyclically permutes them, has order five, and is a modular loop: mutation at the first vertex followed by interchanging labels restores  $\varepsilon$ . The cyclic exchange identities give the same positive integral Laurent formulas in every adjacent cluster chart. This is the standard type  $A_2$  structure also used in [1, Example 4.10].

Take  $G = \langle T \rangle$  and  $\Lambda = \{q_1, q_2, q_3, q_4, q_5, 10\}$ . Its support contains  $\{\pm e_1, \pm e_2\}$ . The exponents of  $q_3$  differ by  $e_2$ , and those of  $q_5$  differ by  $e_1$ . Since  $\ker \varepsilon = 0$ , this proves both filling conditions. On  $[1, 2]^2$  we have

$$q_1, q_2 \leq 2, \quad q_3, q_5 \leq 3, \quad q_4 \leq 5.$$

The invariant maximum is therefore  $\log 10$  on this entire box. It cannot be smaller anywhere because  $10 \in \Lambda$ .  $\square$

**Proposition 2.3.** *Let  $\Lambda$  be any filling set, let  $G$  be finite, and let  $K \subset M$  be a nonempty compact set. A positive integral constant can be adjoined to  $\Lambda$  so that every point of  $K$  minimizes the new maximum potential, while both filling conditions remain valid.*

*Proof.* Choose an integer  $C > 0$  larger than the maximum on  $K$  of every member of the finite set  $\Lambda_G$ . Such an integer exists by continuity and compactness. The function  $C$  is universally positive integral Laurent, has support  $\{0\}$  and zero slope space, and is fixed by  $G$ . Thus  $\Lambda' = \Lambda \cup \{C\}$  is still a filling set. Its maximum potential is  $\max\{L_{\max}, \log C\}$ , which equals  $\log C$  on  $K$  and is at least  $\log C$  everywhere.  $\square$

In positive dimension  $K$  can have nonempty interior. The failure is therefore intrinsic to the stated hypotheses, rather than a degeneracy of one example. In the printed proof of [1, Theorem 4.7], Step 1 transfers strict convexity from an individual member to a maximum; an inactive member cannot justify that transfer. Step 2 infers a positive active slope in a prescribed direction from membership of zero in the convex hull of active slopes. That inference also need not hold. Balance of the full support does not supply the missing condition on the active subset.

### 3 The sum potential

**Theorem 3.1.** *Let  $\Lambda$  be a nonempty finite collection of universally positive Laurent functions on  $M$  with balanced support in one global seed chart. For any finite group  $G$  of cluster modular transformations, the function*

$$H(p) = \sum_{g \in G} \sum_{F \in \Lambda} F(gp) \tag{4}$$

*is  $G$ -invariant, coercive and strictly convex in that logarithmic chart. It has a unique minimizer  $p_*$ , and every element of  $G$  fixes  $p_*$ . No slope-span hypothesis is required.*

*Proof.* Suppose first that  $d > 0$ . In the chosen chart, universal positivity and finiteness give

$$H(u) = \sum_{\alpha \in S} c_\alpha e^{\langle \alpha, u \rangle}, \quad c_\alpha > 0,$$

with  $S$  finite. Positivity prevents cancellation, and the identity summand retains the balanced support from  $\Lambda$ . Hence there is  $\delta > 0$  such that the closed ball  $\overline{B}(0, \delta)$  lies in  $\text{conv } S$ . For each  $u \neq 0$ , maximization of a linear functional on  $\text{conv } S$  gives an  $\alpha \in S$  with  $\langle \alpha, u \rangle \geq \delta \|u\|$ . Consequently

$$H(u) \geq c_{\min} e^{\delta \|u\|}, \quad c_{\min} = \min_{\alpha \in S} c_\alpha > 0. \tag{5}$$

The bound also holds at zero. Thus  $H$  tends to infinity at infinity and, being continuous, attains a global minimum.

For every  $v \neq 0$ ,

$$v^\top \text{Hess } H(u) v = \sum_{\alpha \in S} c_\alpha e^{\langle \alpha, u \rangle} \langle \alpha, v \rangle^2 > 0. \tag{6}$$

Indeed,  $S$  spans  $\mathbb{R}^d$ , so some term is strictly positive. It follows that  $H$  is strictly convex on  $\mathbb{R}^d$  and its minimizer is unique. For  $h \in G$ , right multiplication by  $h$  permutes  $G$ , so

$$H(hp) = \sum_{g \in G} \sum_{F \in \Lambda} F(ghp) = H(p).$$

Therefore  $hp_*$  is also a minimizer, and  $hp_* = p_*$ . For  $d = 0$  the global chart is a single point, so the conclusion is immediate.  $\square$

**Corollary 3.2.** *The function  $L_{\text{sum}} = \log H$  is likewise coercive and strictly convex in the chosen logarithmic chart, and has the same minimizer. If  $m = |G| |\Lambda|$ , then*

$$L_{\text{max}} \leq L_{\text{sum}} \leq L_{\text{max}} + \log m.$$

*Proof.* Set  $w_\alpha = c_\alpha e^{\langle \alpha, u \rangle} / H(u)$ , so all  $w_\alpha$  are positive and sum to one. Direct differentiation gives

$$v^\top \text{Hess}(\log H)(u)v = \sum_{\alpha} w_{\alpha} \langle \alpha, v \rangle^2 - \left( \sum_{\alpha} w_{\alpha} \langle \alpha, v \rangle \right)^2. \quad (7)$$

This variance is zero precisely when  $\langle \alpha, v \rangle$  is constant on  $S$ . For  $v \neq 0$  that would put  $\text{conv } S$  in a proper affine hyperplane, contradicting full-dimensional interior. Coercivity follows by taking logarithms in (5). The final inequalities follow by comparing the largest of the  $m$  positive summands in (4) with their sum; repetitions cause no problem.  $\square$

The bounded difference in Corollary 3.2 does not transfer uniqueness back to  $L_{\text{max}}$ . Nor does uniqueness of the selected potential's minimizer assert that the action has exactly one fixed point. The selected point can depend on  $\Lambda$ .

## 4 Consequences, scope and verification

Every filling set of [1, Definition 4.5] has balanced support. Theorem 3.1 therefore supplies the common fixed point asserted by that paper's Theorem 4.8, without (1). Any equivariant map sends this point to a fixed point; in particular this recovers the conclusion on the image of the cluster ensemble map. This argument is not an independent verification of all support constructions or later corollaries in that paper.

The correction arose while investigating the AIM question on proper cluster modular actions on positive braid varieties [2]. The sum in (4) uses finiteness of  $G$  essentially. We have not constructed such an invariant finite sum for an infinite group, proved properness of its action, or settled the requested braid-centralizer relationship. In particular, the original two-part AIM question is not counted as resolved by this note.

The finite-group fixed-point conclusion is credited to Ishibashi; balanced exponential-sum coercivity and the Hessian formulas are standard convex-analysis ingredients. A bounded literature check did not identify a prior version of this specific correction, but does not establish its absolute priority.

The accompanying standard-library Python programs check the five exchange identities symbolically and evaluate the examples using rational arithmetic. The broader algebra checker passed 16,154 checks and a separate moment-matrix checker passed 205 checks, each in Python 3.13.5 and 3.12.14. The latter includes negative controls showing why mere linear span or relative interior in a proper subspace is insufficient. Replaying a program in a second runtime is not independent

mathematical review. The proofs of the compact-set statement and the general sum theorem are the arguments above, not an extrapolation from these finite tests.

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## References

- [1] T. Ishibashi, *Fixed point theorem for cluster modular groups*, arXiv:2603.14338v2, 2026. <https://arxiv.org/abs/2603.14338v2>.
- [2] American Institute of Mathematics, *Cluster algebras and braid varieties*, problem list, section 8, Problem 1, 2023. <http://aimpl.org/clusterbraid/8/>. Workshop report: <https://aimath.org/pastworkshops/clusterbraidrep.pdf>.