

Exact Heisenberg Determinant Limits and Tower-Dependent Homological Torsion

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2 October 2026

Abstract

Kammeyer recently disproved determinant approximation using $f = 1 - 2a + 2b$ in the integral group ring of the discrete Heisenberg group. We compute the exact finite determinant limit in this example: along the congruence tower modulo powers of any fixed odd prime p , it is $2^{1/(p+1)}$, whereas the infinite Fuglede–Kadison determinant is 2. An exact central-character factorization and an integral-lattice tail bound justify the limit without passing an unbounded logarithm through weak spectral convergence. Attaching a two-sphere and a three-cell to the Heisenberg nilmanifold realizes the same discrepancy in integral homology. The resulting finite three-dimensional CW complex has determinant- L^2 -acyclic universal cover and L^2 -torsion $\log 2$, but its normalized alternating homological torsion limit is $\log 2/(p+1)$. Thus this limit depends on the residual tower for a single space. This refutes the unrestricted finite-CW formulation of modified homological torsion approximation. The space is neither aspherical nor a closed manifold, and no aspherical-manifold conjecture is refuted. The underlying counterexample and representation-theoretic ingredients are explicitly credited; no absolute-priority claim is made.

1 Statements and scope

Let $H = H_3(\mathbb{Z})$ be the integral Heisenberg group, written as triples with

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

Put $a = (1, 0, 0)$, $b = (0, 1, 0)$ and $c = (0, 0, 1)$, so that $ab = bac$ and c is central. For an odd prime p and $i \geq 1$, define

$$H_i(p) = \ker(H \longrightarrow H_3(\mathbb{Z}/p^i\mathbb{Z})), \quad q = p^i, \quad Q_q = H/H_i(p).$$

The subgroups form a descending normal residual chain, and $|Q_q| = q^3$. Let M_q be right multiplication by $f = 1 - 2a + 2b$ on $\mathbb{C}[Q_q]$ and write $D_q = |\det_{\mathbb{C}} M_q|$. The integral matrix M_q is the identity modulo 2, so its determinant is an odd nonzero integer. Its normalized finite Fuglede–Kadison determinant is therefore D_q^{1/q^3} .

Theorem 1. *For every fixed odd prime p ,*

$$\lim_{i \rightarrow \infty} \frac{\log D_{p^i}}{p^{3i}} = \frac{\log 2}{p+1}, \quad \lim_{i \rightarrow \infty} \det_{\text{FK}} M_{p^i} = 2^{1/(p+1)}. \quad (1)$$

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Kammeyer proved that the infinite operator R_f on $\ell^2(H)$ is injective and has Fuglede–Kadison determinant 2, while the finite determinants for $p = 3$ have limsup at most $5^{1/3}$ [4, Theorem 2 and Sections 2–3]. The infinite-determinant calculation uses Deninger’s formula [3, Theorem 11(a)], as applied in [4]. Theorem 1 computes the limit for that published example; it is not a new discovery of the failure of determinant approximation. Boschheidgen’s finite representations and limiting zero-eigenvalue proportion $p/(p + 1)$ for $a - b$ are central antecedents [1, Section 4]. Related binomial determinant factors for $H_3(\mathbb{F}_p)$ appear in [6].

For a finite connected CW complex W , use the *unnormalized* homological statistic

$$\rho^{\mathbb{Z}}(W) = \sum_{j \geq 0} (-1)^j \log |\operatorname{Tor} H_j(W; \mathbb{Z})|. \quad (2)$$

The trivial torsion group has order one. For L^2 -torsion we use the sign convention

$$\rho^{(2)}(C_*) = \sum_{j \geq 0} (-1)^{j+1} \log \det_{\text{FK}}(d_j).$$

Determinant- L^2 -acyclic means that reduced L^2 -homology vanishes and the differentials are of determinant class.

Theorem 2. *There exists a connected finite three-dimensional CW complex Y with $\pi_1(Y) = H$ such that $\tilde{Y}$ is determinant- L^2 -acyclic and*

$$\rho^{(2)}(\tilde{Y}) = \log 2.$$

For every fixed odd prime p , let $Y_i(p) = H_i(p) \backslash \tilde{Y}$. Then

$$\lim_{i \rightarrow \infty} \frac{\rho^{\mathbb{Z}}(Y_i(p))}{[H : H_i(p)]} = \frac{\log 2}{p + 1}. \quad (3)$$

The complex Y is neither aspherical nor a closed manifold.

In particular, the left side of (3) depends on the residual tower of this *same* space. Hughes and Lück formulate modified homological torsion approximation for connected finite determinant- L^2 -acyclic CW complexes in [2, Conjecture 1.2, version 2]. Their notation includes division by the covering index in $\rho^{\mathbb{Z}}(X[i])$; our notation (2) leaves that division explicit. Theorem 2 contradicts that unrestricted finite-CW formulation. Kammeyer already constructs a related free cocompact CW counterexample for a different torsion-approximation statement [4, Theorem 6]; here we verify determinant- L^2 -acyclicity, the actual universal cover, and integral homological torsion.

The assertion for a *closed aspherical* odd-dimensional manifold is a separate problem. It concerns middle-dimensional homological torsion, as in [5, Conjecture 1.12(2)] and [2, Conjecture 1.1]. The investigation began with AIM Problem 11.1, listed as AIM-GEOMETRIC_GROUP_THEORY-0134 in the frozen `ulamai/UnsolvedMath` corpus. Neither that original question nor the aspherical-manifold assertions are settled here.

2 An exact central-order factorization

We calculate with the left regular matrices of Q_q . Inversion of the group basis conjugates these to the transposes of the right regular matrices, so their determinants agree. Thus we may use the relation $AB = \zeta BA$ on a central-character space without ambiguity about right-operator composition.

For m odd and t a positive even integer define

$$B_m(t) = \prod_{\alpha^m = \beta^m = 1} (1 - t\alpha + t\beta). \quad (4)$$

This is the determinant of $I - tR_x + tR_y$ on $\mathbb{Z}[(\mathbb{Z}/m\mathbb{Z})^2]$, hence an integer congruent to 1 modulo 2. It is positive: simultaneous conjugation pairs the nonreal factors into positive squared moduli, and for odd m its only fixed pair is $(\alpha, \beta) = (1, 1)$, whose factor is one.

Proposition 3. *For $q = p^i$ with p odd,*

$$D_{p^i} = \prod_{s=0}^i B_{p^{i-s}} \left(2^{p^s} \right)^{p^s \varphi(p^s)}, \quad \varphi(1) = 1. \quad (5)$$

Proof. Fix a central eigenvalue ζ of exact order $r = p^s$ and put $m = q/r$. Its eigenspace V_ζ in the regular representation has dimension q^2 . On this space, a^r and b^r commute with a, b and with one another. Their simultaneous eigenvalues α, β satisfy $\alpha^m = \beta^m = 1$.

Each joint space $W_{\zeta, \alpha, \beta}$ has dimension r^2 . Indeed, the subgroup $U_r = \langle a^r, b^r, c \rangle$ has order $m^2 q$ and index r^2 in Q_q . The assignments

$$a^r \mapsto \alpha, \quad b^r \mapsto \beta, \quad c \mapsto \zeta$$

define a one-dimensional character of U_r : generator orders are respected, and the commutator c^{r^2} is killed. The restriction of the regular representation to U_r is the direct sum of r^2 copies of its regular representation. Each such copy contains this character exactly once. This proves both the dimension and occurrence of every root pair.

Write A, B for the actions of a, b on the joint space. We have $A^r = \alpha I$, $B^r = \beta I$, and $AB = \zeta BA$. The r distinct roots of $\lambda^r = \beta$ are the eigenvalues of B . The operator A permutes their eigenspaces cyclically. Each consequently has dimension r . Choosing a basis of one of them and taking its A -iterates splits the joint space into r invariant blocks of dimension r . In each block A is a cyclic shift with wrap weight α , and B is diagonal with entries $\lambda \zeta^j$, $0 \leq j < r$, in one of the two cyclic orders.

For such a block,

$$\begin{aligned} \det(I - 2A + 2B) &= \prod_{j=0}^{r-1} (1 + 2\lambda \zeta^j) - 2^r \alpha \\ &= 1 + 2^r \beta - 2^r \alpha. \end{aligned} \quad (6)$$

For $r \geq 2$, the determinant expansion has only the full diagonal and full shift permutations. For $r = 1$ the same equality is scalar. The second equality uses $\prod_j (1 - t\zeta^j) = 1 - t^r$ and oddness of r . There are r blocks for each (α, β) and $\varphi(r)$ central characters of exact order r . Multiplying (6) proves (5). The dimension check is

$$\sum_{s=0}^i \varphi(p^s) q^2 = q^3.$$

□

In particular, the primitive central-order summand has $r = q$, $m = 1$, and $B_1(2^q) = 1$. These are the large unipotent blocks in the published counterexample. Lower central orders cannot simply be discarded when one seeks the exact limit.

3 The logarithmic limit and its tail

Let $E_{i,s}$ be the sum of the central-character spaces of exact order p^s and let $D_{i,s}$ be the absolute determinant of M_{p^i} on it. Then $D_{p^i} = \prod_s D_{i,s}$.

Lemma 4. For $0 \leq s \leq i$,

$$0 \leq \frac{\log D_{i,s}}{p^{3i}} \leq \frac{\varphi(p^s)}{p^i} \log 5. \quad (7)$$

Consequently, for $0 \leq J \leq i$,

$$0 \leq \sum_{s=0}^{i-J} \frac{\log D_{i,s}}{p^{3i}} \leq p^{-J} \log 5. \quad (8)$$

Proof. The subspace $E_{i,s}$ is rational: it is the kernel of $\Phi_{p^s}(R_c)$, where Φ_{p^s} is an integral cyclotomic polynomial. Its intersection with $\mathbb{Z}[Q_q]$ is a full-rank lattice preserved by $T = R_a - R_b$. In an integral basis of this lattice, $M_q = I - 2T$ has odd integer determinant. Thus $D_{i,s} \geq 1$.

The central unitary commutes with M_q and its adjoint, so this subspace is reducing. Its dimension is $\varphi(p^s)q^2$ and the operator norm of M_q is at most $1 + 2 + 2 = 5$. Each singular value of the restriction is at most 5. This proves (7). Finally

$$\frac{1}{p^i} \sum_{s=0}^{i-J} \varphi(p^s) = p^{-J},$$

which proves (8). $\square$

Proof of Theorem 1. Set $j = i - s$, and first fix $j \geq 0$. Then $m = p^j$ is fixed and $r = p^{i-j}$ tends to infinity. In (4), the m pairs with $\alpha = \beta$ contribute one. For each of the other $m^2 - m$ pairs,

$$\log |1 + 2^r(\beta - \alpha)| = r \log 2 + \log |\beta - \alpha| + o(1).$$

For $m > 1$ the minimum root separation is $d_m = 2 \sin(\pi/m) > 0$. Once $2^r d_m \geq 2$, the error for any pair is at most $2/(2^r d_m)$, using $|\log |1 + z|| \leq 2|z|$ for $|z| \leq 1/2$. The derivative of $X^m - 1$ at α gives $\prod_{\beta \neq \alpha} |\beta - \alpha| = m$. Therefore

$$\log B_m(2^r) = (m^2 - m)r \log 2 + m \log m + o(1). \quad (9)$$

For $m = 1$ both sides vanish exactly, apart from the unnecessary error notation.

By Proposition 3 the multiplicity is $r\varphi(r)$. Since $q = rm$ and $\varphi(r)/r = (p-1)/p$ for sufficiently large i ,

$$\lim_{i \rightarrow \infty} \frac{\log D_{i,i-j}}{q^3} = \frac{p-1}{p} (p^{-j} - p^{-2j}) \log 2. \quad (10)$$

The constant term in (9) disappears after normalization; its coefficient is $O(1/r)$ for this fixed m .

Fix J . The finitely many terms $j < J$ converge by (10), and the whole remaining tail is between zero and $p^{-J} \log 5$ by Lemma 4. Taking $i \rightarrow \infty$ first and then $J \rightarrow \infty$ therefore proves convergence of the full normalized logarithm to

$$\begin{aligned} \frac{p-1}{p} \sum_{j=0}^{\infty} (p^{-j} - p^{-2j}) \log 2 &= \frac{p-1}{p} \left(\frac{p}{p-1} - \frac{p^2}{p^2-1} \right) \log 2 \\ &= \frac{\log 2}{p+1}. \end{aligned}$$

Exponentiation gives the normalized determinant limit. $\square$

The tail argument is indispensable. No uniform estimate in growing m was assumed in (9), and weak spectral convergence alone would not justify integration against an unbounded logarithm.

4 A finite CW complex with tower-dependent torsion

Let $N = H \backslash H_3(\mathbb{R})$ be the standard Heisenberg nilmanifold. It is a closed oriented three-manifold with contractible universal cover $H_3(\mathbb{R}) \cong \mathbb{R}^3$. Choose a finite CW structure and a base vertex.

Lemma 5. *For a finite-index subgroup $L \leq H$, let its image in $\mathbb{Z}^2$ have index d , and write $L \cap \langle c \rangle = \langle c^r \rangle$. Then $r \mid d$, $[H : L] = dr$, and*

$$L_{\text{ab}} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/(d/r).$$

Proof. Choose an oriented lattice basis of the image and lifts $u, v \in L$. Their commutator is c^d . With $w = c^r$, every element of L has a unique collected expression $u^m v^n w^\ell$: projection determines m, n , and the remaining central element determines ℓ . Thus

$$L = \langle u, v, w \mid [u, w] = [v, w] = 1, [u, v] = w^{d/r} \rangle.$$

Here the relation first shows $r \mid d$, and uniqueness of collection shows that no further relation was introduced. Abelianization gives the assertion. The preimage of the projected lattice has index d in H and is $L\langle c \rangle$; its index over L is r . $\square$

For $L = H_i(p)$, its projected index is q^2 and central index is q . Hence the corresponding cover $N_i(p)$ has

$$H_0(N_i; \mathbb{Z}) = H_3(N_i; \mathbb{Z}) = \mathbb{Z}, \quad H_1(N_i; \mathbb{Z}) = \mathbb{Z}^2 \oplus \mathbb{Z}/q, \quad H_2(N_i; \mathbb{Z}) = \mathbb{Z}^2. \quad (11)$$

The last equality follows from integral Poincaré duality and $H^1(N_i; \mathbb{Z}) = \text{Hom}(H_1(N_i; \mathbb{Z}), \mathbb{Z})$. The lemma also gives the sharp general bound $|\text{Tor } L_{\text{ab}}| = [H : L]/r^2 \leq [H : L]$; thus the normalized logarithmic torsion of nilmanifold covers tends to zero whenever their degrees tend to infinity. This is consistent with the known amenable-normal-subgroup case, not a new counterexample for N .

Proof of Theorem 2. Form $Z = N \vee S^2$. Its fundamental group is H . The universal cover is the contractible cover of N with one sphere at every lifted wedge point, so Hurewicz identifies $\pi_2(Z)$ with a free $\mathbb{Z}H$ -module on the new sphere. Attach one three-cell along the element $f = 1 - 2a + 2b$ of that module to obtain Y .

More explicitly, a pinched sphere with degrees $1, -2, 2$, transported along whiskers representing $1, a, b$ in the old one-skeleton, realizes this attaching map. It has zero cellular degree on every old two-cell of N . The new two-cell has zero boundary, and the old attaching maps stay in N . Therefore, with compatible lifts, there is an actual based chain decomposition

$$C_*(\tilde{Y}; \mathbb{Z}) = C_*(\tilde{N}; \mathbb{Z}) \oplus [\mathbb{Z}H \xrightarrow{R_f} \mathbb{Z}H]_{3 \rightarrow 2}. \quad (12)$$

Both mixed blocks vanish by this construction. Adding these cells does not change the fundamental group. The decomposition persists on every finite cover associated with $H_i(p)$.

The finite matrix M_q is nonsingular over $\mathbb{Q}$, as already noted. Its integral kernel is zero and its cokernel has order D_q . Combining (12) with (11) gives

$$\begin{aligned} H_0(Y_i; \mathbb{Z}) &= H_3(Y_i; \mathbb{Z}) = \mathbb{Z}, \\ H_1(Y_i; \mathbb{Z}) &= \mathbb{Z}^2 \oplus \mathbb{Z}/q, \\ H_2(Y_i; \mathbb{Z}) &= \mathbb{Z}^2 \oplus \text{coker } M_q. \end{aligned} \quad (13)$$

In particular,

$$\rho^{\mathbb{Z}}(Y_i(p)) = \log D_q - \log q. \quad (14)$$

Theorem 1 and $\log q/q^3 \rightarrow 0$ yield (3).

For the infinite complex, Lück's theorem [5, Corollary 1.13] applies to N , whose fundamental group has the nontrivial infinite cyclic normal subgroup $\langle c \rangle$. Thus $\tilde{N}$ is determinant- L^2 -acyclic and has L^2 -torsion zero. We use this published theorem, not an inference from the finite torsion limit. Kammeyer's injectivity and determinant calculation for R_f give a determinant-class map with determinant 2 [4]. A square injective operator over a finite von Neumann algebra has dense image: polar decomposition equates its kernel and adjoint-kernel dimensions, and the trace is faithful. Thus the two-term summand in (12) is L^2 -acyclic. Its degree-three differential contributes $+\log 2$, proving $\rho^{(2)}(\tilde{Y}) = \log 2$ by direct-sum additivity.

It remains to establish the geometric exclusions, not merely leave them unchecked. At $q = 3$ formula (5) gives

$$D_3 = B_3(2) = 1729 = 7 \cdot 13 \cdot 19. \quad (15)$$

If $R_f : \mathbb{Z}H \rightarrow \mathbb{Z}H$ were onto, some u would satisfy $uf = 1$. Its reduction would give an integral one-sided inverse to M_3 , forcing its determinant to be ± 1 . Equation (15) rules this out. Hence $H_2(\tilde{Y}; \mathbb{Z}) = \text{coker } R_f \neq 0$, and Y is not aspherical.

If Y were a closed three-manifold, its connected 27-sheeted cover $Y_1(3)$ would be one as well. Its $H_3 = \mathbb{Z}$ forces orientability, which makes H_2 torsion-free by integral Poincaré duality. But (13) and (15) give nonzero H_2 torsion of order 1729. This contradiction proves that Y is not a closed manifold. $\square$

The additional cells produce the discrepancy and simultaneously destroy the hypotheses of the closed-aspherical question. Replacing Y by an aspherical space with the same fundamental group does not preserve (12). For N itself, the middle-dimensional torsion formula is the known equality $0 = 0$.

5 Exact checks and reproducibility

Two standard-library Python checkers accompany this note. They use explicit exceptions, so optimized Python does not disable the checks. The first computes finite subgroup controls and the actual 27 by 27 right-regular matrix, obtaining (15). It also verifies the primitive-central nilpotence block and compares modular determinants. The second independently eliminates the full regular matrices for $q = 3$ and $q = 5$ and matches Proposition 3:

$$D_3 = 1729, \quad D_5 = 3277440001.$$

For example, the $q = 3$ abelian block also gives, with $\omega^3 = 1$ and $\omega \neq 1$,

$$B_3(2) = 7(-11 - 18\omega)(-11 - 18\omega^2) = 1729.$$

At $q = 9$, the checker evaluates the factorization, rather than claiming a separate 729 by 729 elimination:

$$B_9(2) = 1836032914900504688395761745921, \quad B_3(8) = 7077889, \quad B_1(512) = 1,$$

and $D_9 = B_9(2)B_3(8)^6$.

Oddness is necessary for the displayed factorization. At $q = 2$ the actual signed determinant is -735 , whereas inserting the odd-order formula incorrectly gives -15 . For even r the root product in (6) changes sign. We make no assertion about a $p = 2$ limit. Finite checks support the displayed algebra; the all-level representation argument and tail estimate, not sampling, prove the limit.

This note was developed with AI assistance in literature search, symbolic derivation, code and exposition, and was self-audited in that workflow. It is an unrefereed preprint, not a proof-assistant-certified or independently human-reviewed result. The bounded literature review did not locate this exact fixed-prime limit, but does not certify novelty or absolute priority. The published inputs identified above remain essential credited ingredients.

References

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