

An Explicit Generating Pair for Radu's Simple Tree Lattice

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Abstract

We give an explicit generating pair for the torsion-free infinite simple group in Corollary IV(ii) of Radu's paper on lattices in products of trees. In the notation of that presentation, the pair is x_1x_3, x_1y_3 . Only the first five displayed equations are needed: a short group calculation expresses every original generator in this pair. In particular, the minimum generator number of this specific group is two. We also give a two-generator presentation with seven defining equations. This excludes one concrete candidate for a finitely presented simple group that is not two-generated; it does not resolve the general existence question.

1 The group and its generating pair

Radu constructs a torsion-free infinite simple group $G \cong F_3 *_{F_{11}} F_3$ as an index-four subgroup of a $(4, 5)$ -group [3, Corollary IV(ii)]. It is identified as $\Gamma_{4,5,9}^+$ in the proof of that corollary. The displayed presentation uses the symbols $x_1, x_2, x_3, y_1, y_2, y_3$, with $x_1 = y_1$, and hence gives a presentation on five generators. A presentation's generator count is an upper bound, not necessarily the minimum generator number.

Theorem 1. *For the group G in [3, Corollary IV(ii)],*

$$G = \langle x_1x_3, x_1y_3 \rangle.$$

Consequently its minimum generator number is exactly two.

The AIM list on amenability of discrete groups asks whether there exists a finitely presented simple group that is not two-generated [1, Problem 5.2]. The question is also stated in [2, Question 6.1]. Theorem 1 concerns the single presentation above. It neither answers that existence question nor asserts two-generation for all simple lattices in products of trees. The simplicity and geometric properties used here are Radu's results.

2 An elementary recovery identity

Lemma 2. *Let a, c, e be elements of a group, put*

$$b = ce^{-1}aec^{-1}, \quad d = ec^{-1}ace^{-1},$$

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and suppose that

$$c^2 = e^2, \quad b^2 = da^{-1}d. \quad (1)$$

Set

$$u = ac, \quad v = ae, \quad s = u^{-1}v, \quad t = uv^{-1}.$$

Then

$$a = s^{-2}ts^{-1}t^2, \quad c = a^{-1}u, \quad e = a^{-1}v. \quad (2)$$

In particular, $\langle a, c, e \rangle = \langle u, v \rangle$.

Proof. The first equation in (1) gives

$$s = c^{-1}e = ce^{-1}, \quad t = asa^{-1}.$$

Here $c^{-1}e = ce^{-1}$ follows by multiplying $c^2 = e^2$ on the left by c^{-1} and on the right by e^{-1} . Thus

$$b = sas^{-1}, \quad d = s^{-1}as.$$

The second equation in (1) becomes

$$sa^2s^{-1} = s^{-1}asa^{-1}s^{-1}as.$$

Using $t = asa^{-1}$, this is

$$sat^{-1}a = s^{-1}ts^{-1}ta. \quad (3)$$

Cancel a on the right, multiply by s^{-1} on the left and by t on the right. This gives the first identity in (2). The other two follow from the definitions of u, v . All three original elements are therefore words in u, v , and the reverse inclusion is immediate. $\square$

Proof of Theorem 1. The first five equations in Radu's presentation are

$$\begin{aligned} x_1 &= y_1, & x_2^2 &= y_2y_1^{-1}y_2, & x_3^2 &= y_3^2, \\ x_3^{-1}x_1x_3 &= y_3^{-1}y_2y_3, & x_3^{-1}x_2x_3 &= y_3^{-1}y_1y_3. \end{aligned} \quad (4)$$

Put $a = x_1 = y_1$, $c = x_3$ and $e = y_3$. Equations (5) give

$$x_2 = ce^{-1}aec^{-1}, \quad y_2 = ec^{-1}ace^{-1}.$$

Thus $G = \langle a, c, e \rangle$, and (4) supplies both hypotheses of Lemma 2. The pair in that lemma is precisely $u = x_1x_3$, $v = x_1y_3$. The other six equations in the source presentation are not needed for this generation argument.

Finally, G is infinite and simple by Radu's theorem. An infinite cyclic group is not simple, so G cannot be generated by one element. $\square$

For an explicit word in the generating pair, (2) reads

$$a = v^{-1}uv^{-1}u^2v^{-2}u^2v^{-1}uv^{-1}. \quad (6)$$

This has length 12 in $u^{\pm 1}, v^{\pm 1}$. The words $a^{-1}u$ and $a^{-1}v$ for c, e each have length 13. Conjugating a by s and s^{-1} recovers x_2 and y_2 , respectively.

3 A two-generator finite presentation

The explicit recovery also eliminates one of the two equations in (1). This gives the following presentation without an assertion that the number of relators is minimal.

Proposition 3. *In the free group on u, v , define the words*

$$\begin{aligned} S &= u^{-1}v, & T &= uv^{-1}, & A &= S^{-2}TS^{-1}T^2, \\ C &= A^{-1}u, & E &= A^{-1}v, & B &= SAS^{-1}, & D &= S^{-1}AS. \end{aligned}$$

Then G is presented by the generators u, v and the following seven equations, in which capital letters denote these words, not additional generators:

$$C^2 = E^2, \tag{7}$$

$$B^{-1}AB = D^{-1}A^{-1}D, \tag{8}$$

$$B^{-1}C^{-2}B = D^{-1}AE^{-2}D, \tag{9}$$

$$B^{-1}C^{-1}B^{-1}ACB = D^{-1}AE^{-1}A^{-1}DEA^{-1}D, \tag{10}$$

$$B^{-1}C^{-1}ABCB = D^{-1}AE^{-1}DE^{-1}AEA^{-1}D, \tag{11}$$

$$B^{-1}C^{-1}B^2CB = D^{-1}AE^{-1}AE^{-1}AEA^{-1}D, \tag{12}$$

$$B^{-1}C^{-1}B^{-1}CBCB = D^{-1}AE^{-1}A^{-1}E^{-1}AEA^{-1}D. \tag{13}$$

Proof. Let P be the group with this presentation. In P , equation (7) implies

$$S = C^{-1}E = CE^{-1}, \quad T = ASA^{-1}.$$

Since $A = S^{-2}TS^{-1}T^2$ by definition, reversing the operations in (3) gives

$$B^2 = DA^{-1}D.$$

Therefore the assignment

$$x_1, y_1 \mapsto A, \quad x_2 \mapsto B, \quad x_3 \mapsto C, \quad y_2 \mapsto D, \quad y_3 \mapsto E$$

satisfies the first five equations of Radu's presentation. Equations (8)–(13) are exactly its remaining six equations under this assignment. Hence it defines a homomorphism $G \rightarrow P$.

Conversely, sending u to x_1x_3 and v to x_1y_3 defines a homomorphism $P \rightarrow G$: Lemma 2 recovers the original generators, so all seven equations hold. The composites fix the original generators of G and fix u, v , since $AC = u$ and $AE = v$. The homomorphisms are inverse. $\square$

4 Verification and scope

The generating pair was located by a bounded coset search. A separate coset-equality certificate and a free-word certificate then recovered the expressions in (2). The latter has 40 straight-line instructions; its checker uses only free cancellation. These computations are supplementary: Lemma 2 and the two explicit inverse homomorphisms give proofs independent of the search software.

For a compact exact check of the main algebra, work in the free group on a, s and set $t = asa^{-1}$, $b = sas^{-1}$, $d = s^{-1}as$. Free cancellation verifies

$$b^2(da^{-1}d)^{-1} = s(a(s^{-2}ts^{-1}t^2)^{-1})s^{-1}.$$

Thus replacing the second equation of (1) by the recovery formula is reversible once $t = asa^{-1}$ is known.

The present result determines the rank of one explicit simple lattice; it does not construct a simple group requiring more than two generators. No absolute priority claim is made. This is an unrefereed note. AI assistance was used in computation, algebraic exploration, drafting and verification; the author is responsible for the manuscript.

References

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